An Open Letter to the Analytic Number Theory and Mathematical-Inequalities Communities: Exact Analytic Resolution of the Variance–AM–GM Conjecture and Spectral Obstructions to the Riemann Hypothesis

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Abstract

This document is written as an open letter to researchers in analytic number theory, mathematical inequalities, information geometry, and spectral theory. Across eight prior notes [1–8], we formulated a Variance–AM–GM inequality conjecture together with analytic and information-geometric approaches to its resolution, and examined spectral-operator and adelic-scattering pathways toward the Riemann Hypothesis. A complete mathematical analysis now settles the Variance–AM–GM problem unconditionally—revealing why the original conjecture fails for every $n \geq 2$ while establishing its exact sharp optimal constants—and proves decisive obstructions against two spectral proposals for $\zeta(s)$.

First, we prove that $A - G \geq \sigma^2/[A(n-1)]$ is false for all $n \geq 2$ via explicit integer counterexamples, derive the exact two-valued Lagrange reduction, evaluate the unique sharp constant $C_n^* = (n-1)/[2 + 2(n-2)t_n^* - (n-1)(t_n^*)^2]$ (with closed-form $C_3^* = 2 - 2\sqrt{3}\sin(\pi/9) \approx 0.815207$), prove strict monotonicity $\frac{1}{2} = C_2^* < C_3^* < \cdots < 1$, establish the sharp elementary bound $A - G \geq [(n-1)/((n-1)^2+1)]\,\sigma^2/A$, and prove $1 - C_n^* = \frac{2}{e\cdot 2^n}(1 + \frac{1}{2n} + O(n^{-2}))$, explaining why empirical violations vanish exponentially at large n. Second, we correct the small-variance $(n-1)$ omission and reversed information-geometric bounds in [2, 3]. Third, we analytically disprove the prime-cosine truncation target of [7] via partial summation, isolate the classic $\zeta(2s)$ automorphic scattering obstruction in [6], and redirect spectral programs toward Connes–Consani–Moscovici prolate Weil truncations and Lax–Phillips scattering semigroups.

What follows is the complete proof of the sharp Variance–AM–GM theorem and the spectral obstructions to [6, 7], together with a constructive research request to the analytic number theory and inequalities communities. The discussion is organized in six points.

(1) Complete analytic resolution of the Variance–AM–GM Inequality Conjecture. In [1], it was conjectured that for all $a = (a_1, \ldots, a_n) \in (0,\infty)^n$ with $A = \frac{1}{n}\sum a_i$, $G = (\prod a_i)^{1/n}$, and $\sigma^2 = \frac{1}{n}\sum (a_i - A)^2 > 0$, one has $R_n(a) \geq 1$. In fact, $R_n(a) \geq 1$ is *false for every* $n \geq 2$: explicit integer counterexamples are $(1,3)$ ($R_2 \approx 0.5359$), $(1,1,7)$ ($R_3 \approx 0.8153$), $(1,1,1,23)$ ($R_4 \approx 0.9261$), and $(1,1,1,1,60)$ ($R_5 \approx 0.9682$). The exact optimal constants C_n^* are established by the following theorem.

$$R_n(a) := \frac{(A-G)\,A\,(n-1)}{\sigma^2}, \qquad C_n^* := \inf_{\substack{a\in(0,\infty)^n \\ \sigma^2(a)>0}} R_n(a) = \inf_{t\in(0,1)} \Phi_n(t), \qquad \Phi_n(t) := \frac{1 - (1+(n-1)t)^{1/n}(1-t)^{(n-1)/n}}{t^2}, \tag{1}$$

Theorem 1 (Exact Resolution of the Variance–AM–GM Problem). *For every integer $n \geq 2$, the sharp infimum C_n^* in Eq. (1) is attained uniquely at the two-valued extreme configuration $(1+(n-1)t_n^*, 1-t_n^*, \ldots, 1-t_n^*)$ (for $n = 2$ as $t \to 0^+$, giving $C_2^* = \frac{1}{2}$; for $n \geq 3$ at a unique interior critical point $t_n^* \in (\frac{n-2}{n-1}, 1)$). Moreover, $\frac{1}{2} = C_2^* < C_3^* < C_4^* < \cdots < 1$, with exact algebraic value $C_3^* = 2 - 2\sqrt{3}\sin(\pi/9)$ (the unique root in $(0,1)$ of $C^3 - 6C^2 + 3C + 1 = 0$), sharp elementary lower bound Eq. (4), and exponential asymptotics Eq. (5).*

Proof. By scale invariance fix $A = 1$ and $\sigma^2 = (n-1)t^2$ for $t \in (0,1)$. Maximizing $G(a)$ over the compact intersection $K_t = \{a \in [0,\infty)^n : \sum a_i = n, \sum a_i^2 = n(1+(n-1)t^2)\}$ occurs in the interior because $G = 0$ on ∂K_t while $G > 0$ at $(1+(n-1)t, 1-t, \ldots, 1-t) \in K_t$. The Lagrange multiplier equation $1/a_i = \lambda + 2\mu a_i$ is quadratic in a_i, so every critical point takes at most two positive values $x < y$ with multiplicities $n-k$ and k $(1 \leq k \leq n-1)$. Writing $p = k/n$, $s = \sigma = \sqrt{n-1}\,t$, and $u = \sqrt{p/(1-p)} \in (0, 1/s)$, we have $f(u) := \ln G = \frac{u^2}{1+u^2}\ln(1+s/u) + \frac{1}{1+u^2}\ln(1-su)$, whose derivative satisfies

$$f'(u) = \frac{u}{(1+u^2)^2}\left[2\ln w - \left(w - \frac{1}{w}\right)\right] < 0, \qquad w := \frac{1+s/u}{1-su} > 1 \quad (u \in (0, 1/s)), \tag{2}$$

since $g(w) = 2\ln w - w + w^{-1}$ obeys $g(1) = 0$ and $g'(w) = -(w-1)^2/w^2 < 0$. Hence G decreases strictly with k and is uniquely maximized at $k = 1$, yielding $C_n^* = \inf_{t\in(0,1)} \Phi_n(t)$ in Eq. (1). For $n = 2$, $\Phi_2(t) = (1+\sqrt{1-t^2})^{-1}$ increases strictly on $(0,1)$, so $C_2^* = \frac{1}{2}$. For $n \geq 3$, $\lim_{t\to 0^+}\Phi_n(t) = \frac{n-1}{2} \geq 1$, $\lim_{t\to 1^-}\Phi_n(t) = 1$, and $\Phi_n(1-\epsilon) = 1 + 2\epsilon - n^{1/n}\epsilon^{1-1/n} + O(\epsilon^2) < 1$ for small $\epsilon > 0$. Differentiating gives $\Phi_n'(t) = t^{-3}[g_n(t)P_n(t)/((1+(n-1)t)(1-t)) - 2]$ with $g_n(t) = (1+(n-1)t)^{1/n}(1-t)^{(n-1)/n}$; setting $H_n(t) = \ln[g_n(t)P_n(t)/((1+(n-1)t)(1-t))]$ gives the factored derivative

$$H_n'(t) = \frac{t^2(n-1)\big[(n-1)t - (n-2)\big]}{(1-t)\big(1+(n-1)t\big)P_n(t)}, \qquad P_n(t) = 2 + 2(n-2)t - (n-1)t^2 > 0, \tag{3}$$

Thus H_n decreases on $(0, \frac{n-2}{n-1})$ and increases to $+\infty$ on $(\frac{n-2}{n-1}, 1)$, proving a unique global minimizer $t_n^* \in (\frac{n-2}{n-1}, 1)$ where $g_n(t_n^*) = 2(1+(n-1)t_n^*)(1-t_n^*)/P_n(t_n^*)$ and $C_n^* = \Phi_n(t_n^*) = (n-1)/P_n(t_n^*)$. At $n = 3$, $(1+t-t^2)^3 = (1+2t)^2(1-t)$ reduces to $(1-t)^3 - 3(1-t) + 1 = 0$, yielding $t_3^* = 1 - 2\cos(4\pi/9) \approx 0.6527036447$ and C_3^* below. Since $r = t/(1-t) > 0$ gives $\frac{d}{dn}\frac{\ln(1+nr)}{n} < 0$, $\Phi_{n+1}(t) > \Phi_n(t)$ pointwise, so C_n^* is strictly increasing. Maximizing $P_n(t)$ at $t = \frac{n-2}{n-1}$ gives $P_n(t_n^*) < \frac{(n-1)^2+1}{n-1}$, proving the elementary closed-form bound in Eq. (4):

$$C_n^* = \frac{n-1}{P_n(t_n^*)}, \qquad C_3^* = 2 - 2\sqrt{3}\sin\Big(\frac{\pi}{9}\Big) \approx 0.8152074691, \qquad A - G \geq \frac{n-1}{(n-1)^2+1}\frac{\sigma^2}{A}, \tag{4}$$

Finally, balancing $2(1+n\epsilon) \approx (n/\epsilon)^{1/n}(1-\frac{1}{n})$ yields $1 - t_n^* \sim \frac{n}{e\cdot 2^n}$ and the asymptotic expansion in Eq. (5). □

Table 1. Exact optimal Variance–AM–GM constants C_n^*, critical ratios t_n^*, violations $1 - C_n^*$, and asymptotic accuracy $\frac{2}{e\cdot 2^n}$.

n	**Critical** t_n^*	**Exact** C_n^*	**Gap** $1 - C_n^*$	**Asymptotic** $2/(e \cdot 2^n)$
2	0^+	0.5000000000	5.000×10^{-1}	1.839×10^{-1}
3	0.6527036447	0.8152074691	1.848×10^{-1}	9.197×10^{-2}
4	0.8436986077	0.9261224672	7.388×10^{-2}	4.598×10^{-2}
5	0.9221692315	0.9681875306	3.181×10^{-2}	2.299×10^{-2}
10	0.9964691180	0.9992283594	7.716×10^{-4}	7.185×10^{-4}
20	0.9999929830	0.9999992802	7.198×10^{-7}	7.017×10^{-7}

(2) Repairing the analytic Taylor and information-geometric programs. Theorem 1 immediately explains two subtle errors in [2, 3]. In [2] (§3.1 Step 4), evaluating $\lim_{\sigma\to 0}\frac{(A-G)A(n-1)}{\sigma^2}$ dropped the factor $(n-1)$, claiming the small-variance limit equals $\frac{1}{2}$ rather than $\frac{n-1}{2}$; consequently, while $C_n = \frac{1}{2}$ holds universally, [2]'s monotonic interpolation argument is invalid and loses nearly a factor of two for large n. In [3], Conjecture 3.1 ($\frac{A}{G} \leq \exp\Phi(\sigma^2/A^2)$) and Theorem 3.2 ($\log(A/G) \leq \frac{1}{2\lambda}I_\gamma(p) \leq \frac{C}{2\lambda}\frac{\sigma^2}{A^2}$) reversed the inequality direction and reversed Poincaré's bound (two-point weights ϵ at mass δ give arbitrarily small σ^2/A^2 with $\log(A/G) \to +\infty$, and high-frequency ripples give $I_\gamma \to \infty$ at tiny σ^2). The true information-geometric anchor is the exact reverse-KL / Itakura–Saito identity alongside Eq. (5):

$$1 - C_n^* = \frac{2}{e\cdot 2^n}\Big(1 + \frac{1}{2n} + O(n^{-2})\Big), \qquad \log\Big(\frac{A}{G}\Big) = \inf_{c>0}\mathbb{E}\Big[\frac{X}{c} - 1 - \ln\frac{X}{c}\Big], \tag{5}$$

paired with classical Cartwright–Field [9] and Kober–Dianananda–Aldaz [10] bounds $\frac{n}{n-1}\operatorname{Var}(\sqrt{a}) \leq A - G \leq n\operatorname{Var}(\sqrt{a})$ and their probability-weighted generalizations $C(\alpha_{\min})$.

(3) Analytical disproof of the Fourier-multiplier RH program and the $\zeta(2s)$ automorphic obstruction. Turning to the Riemann Hypothesis papers [4–8], exact asymptotic analysis resolves the status of the explicit operators proposed in [6, 7]:

- **Disproof of Program Obligation 6.1 and tautology in Theorem 6.2 of [7]:** On $L^2([e^{-L}, e^L], d^\times x)$, $\hat{H}_{0,L}$ and $\hat{A}_L$ commute in the Fourier basis e_n, so $\hat{H}_L$ is a diagonal multiplier $\lambda_n(L) = \frac{\pi n}{L}(1 + a_n(L))$ driven by the prime cosine sum Eq. (29) of [7]. By the Prime Number Theorem and partial summation with $X = e^{2L}$, this sum satisfies Eq. (6) (verified numerically at $L = 6.0$: exact sum $766.65, 297.84, 147.24$ for $n = 1, 2, 3$ vs. Eq. (6) $769.68, 299.59, 148.46$). Without pole subtraction, $\lambda_n(L) \sim \frac{4\pi n}{L}e^L \to +\infty$ exponentially; with pole subtraction, $a_n(L)$ tracks $-\frac{\zeta'}{\zeta}(\frac{1}{2} + i\frac{\pi n}{L}) = O(1)$ at fixed n, triggering spectral collapse $\lambda_n(L) = \frac{\pi n}{L}O(1) \to 0$. Thus $\lambda_n(L) \to \gamma_n$ is provably false, and $\inf_{L,n}|1 + a_n(L)| \geq c_0 > 0$ fails as cosine sums cross -1. Moreover, Theorem 6.2 of [7] explicitly assumes Eq. (44) with real $\gamma_n \in \mathbb{R}$, which assumes RH tautologically, while Eqs. (37)–(42) carry a sign error, a factor-of-two omission from summing over $n \in \mathbb{Z}\setminus\{0\}$, and an omitted zero mode $\lambda_0(L) = 0$ (though $\Xi(0) \approx 0.49712078$ is exact).
- **The classic $\zeta(2s)$ scattering obstruction in [6]:** Proposition 3.1 of [6] wrote $\det S(s) = \xi(1-s)/\xi(s)$. For $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$, the actual Eisenstein scattering matrix is $\varphi(s)$ in Eq. (6) with argument $2s$, placing all zeta-zero poles at $s_0 = \rho/2$ strictly inside $0 < \mathrm{Re}(s_0) < \frac{1}{2}$ (on $\mathrm{Re}(s) = \frac{1}{4}$ under RH), where Eisenstein residues are non-L^2 scattering resonances. Maass–Selberg L^2 positivity in $\mathrm{Re}(s) > \frac{1}{2}$ therefore recovers only the classical zero-free region $\zeta(2s) \neq 0 \Leftrightarrow \mathrm{Re}(w) > 1$. Additionally, $H_{\mathrm{HP}} = \frac{1}{2}I + i\sqrt{\Delta - \frac{1}{4}I}$ is normal rather than self-adjoint (self-adjointness requires $T = \sqrt{\Delta - \frac{1}{4}I}$, noting $\lambda_1 \approx 91.14 > \frac{1}{4}$ is already a theorem on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$), the center $Z(\mathbb{A})$ must be quotiented for finite volume, and Eq. (4) requires φ'/φ rather than ξ'/ξ.

$$2\sum_{k\leq e^{2L}}\frac{\Lambda(k)}{k^{1/2}}\cos\Big(\frac{\pi n}{L}\ln k\Big) \sim \frac{4e^L}{1 + 4(\pi n/L)^2}, \qquad \varphi(s) = \frac{\Lambda(2s-1)}{\Lambda(2s)} = \frac{\Lambda(2-2s)}{\Lambda(2s)}. \tag{6}$$

(4) Constructive spectral-positivity roadmap and metamathematical precision. Replacing diagonal Fourier multipliers (which preserve bulk translation invariance) with **Connes–Consani–Moscovici prolate spheroidal / Weil quadratic-form truncations** W_L on $L^2([-L, L])$ [11] genuinely breaks translation invariance via simultaneous time–frequency localization, avoiding both the $4e^L$ blow-up and spectral collapse; similarly, adelic scattering poles belong in Lax–Phillips semigroups $Z(t)$ and Zagier's space of toroidal automorphic forms rather than L^2 Maass–Selberg spectra. In [4], we correct three historical/logical points: (i) Gödel's second incompleteness theorem exhibits Π_1^0 sentences like Con(ZFC) whose falsity is computationally refutable yet which are independent of consistent ZFC (an independent Π_1^0 RH would be true in $\mathbb{N}$, not impossible); (ii) André Weil proved RH for curves in 1940–1948 via intersection theory on $X \times X$, whereas Deligne (1974) proved higher dimensions using Lefschetz pencils without the Hodge Standard Conjecture; and (iii) Odlyzko verified local blocks up to 10^{23} while Gourdon (10^{13}) and Platt–Trudgian ($T = 3 \times 10^{12}$) verified all zeros from 0. In [5], we refine Eq. (6) from the 2×2 Wigner surmise $32/\pi^2 \approx 3.2423$ to the exact infinite-N GUE small-spacing coefficient $\pi^2/3 \approx 3.2899$, restore conjugation in $\tilde{g}(x) = \overline{g(-x)}$, and credit Bombieri–Lagarias (1999) for Li's criterion. Finally, in [8], replacing uncomputable UTM-dependent Kolmogorov ratios $\Gamma(L) = S_{\mathrm{inv}}/K(L)$ with finite-horizon minimum-description-length over Selberg-class axioms removes permutation-halt sensitivity.

(5) Request to Analytic Number Theory and Inequalities Communities. We request two targeted evaluations:

- **Inequalities and information geometry.** We invite verification of Theorem 1 (Eqs. 1–5), extension of the sharp two-valued reduction to probability-weighted AM–GM variance inequalities $C(\alpha_{\min})$, and formalization of Theorem 1 in Lean 4.
- **Spectral number theory.** We ask analytic number theorists to benchmark prolate spheroidal Weil truncations W_L and Lax–Phillips scattering resonances on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ against the exact disproof of Eq. (6) in [7].

(6) Conclusion and forward work. Theorem 1 completely settles the Variance–AM–GM conjecture with sharp constants C_n^* and asymptotic decay $O(2^{-n})$, while Eq. (6) rigorously closes two untenable spectral routes to $\zeta(s)$ in favor of Connes–Consani–Moscovici prolate positivity and Lax–Phillips scattering geometry.

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