Beyond Electric-Dipole Coupling:

Optical Zeeman Torque, Floquet Gauge Dynamics, and Topological State Control

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Abstract

Recent experiments establish that optical-frequency magnetic fields ($\mathbf{B}_{\text{opt}}$) exert measurable torque on ordered moments in magneto-optical media, while helicity-dependent optical drives induce non-thermal switching of magnetic and Chern domains in twisted van der Waals heterostructures. Rather than heralding an unconstrained breakdown of classical electrodynamics, these findings expose specific physical regimes where collective spin coherence and non-trivial band geometry alter the conventional light-matter coupling hierarchy.

In this perspective, written for our website, we formulate the microscopic conditions under which optical Zeeman driving couples to non-equilibrium topological invariants. Evaluating the high-frequency Magnus expansion for a generic spin-orbit-coupled band structure, we demonstrate that cross-coupling between the vector potential and the optical magnetic field is governed by the spin-dependent velocity operator $\hat{\mathbf{v}}_{\text{spin}} = \frac{1}{\hbar}\nabla_{\mathbf{k}}\mathbf{d}(\mathbf{k})\cdot\boldsymbol{\sigma}$, generating a manifestly Hermitian secular Zeeman field at order $\mathscr{O}(\Omega^{-1})$. We formulate a dimensionless operator ratio $\mathscr{R}_{\text{mix}}(\mathbf{k}) \equiv \|\mathscr{H}^{(1)}_{\text{mixed}}\|/\|\mathscr{H}^{(1)}_{\text{electric}}\|$, showing that quenching the scalar kinetic dispersion ($v_0 \to 0$) does not eliminate electric interactions, but isolates the spin-dependent sector where mixed optomagnetic terms compete with electric driving.

We delineate two distinct interaction channels: Channel I (Zeeman-Floquet driving, operating in standard transverse plane waves where $\mathbf{E}_{\text{opt}}\cdot\mathbf{B}_{\text{opt}} = 0$) and Channel II (pseudoscalar electrodynamics, requiring structured fields where $\mathbf{E}_{\text{opt}}\cdot\mathbf{B}_{\text{opt}} \neq 0$). For Channel I, we establish the topological stability criteria in the quasi-Floquet adiabatic regime, proving that coherent state reconfiguration requires independent closure of the 0- or π-quasienergy gaps. Finally, by integrating out the electronic degrees of freedom in a fully regularized two-dimensional lattice model, we derive the adiabatic Chern–Simons response and formulate its non-equilibrium Keldysh-Floquet extension, providing a concrete theoretical framework to distinguish coherent Floquet gap inversion from exciton-polaron-mediated switching in moiré Chern ferromagnets.

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1 The Multipole Expansion and Dual Dimensionless Ratios

In single-particle quantum mechanics, the coupling of an electron to an electromagnetic field is described by minimal coupling alongside the Pauli-Zeeman interaction:

$$\hat{\mathscr{H}} = \frac{1}{2m_e}\left(\hat{\mathbf{p}} + e\mathbf{A}(\mathbf{r},t)\right)^2 + V(\mathbf{r}) + \mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_{\text{opt}}(\mathbf{r},t), \tag{1}$$

where $\mu_B = e\hbar/(2m_e)$ and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. For a transverse plane wave in vacuum ($E_0 = cB_0$), the ratio of the Zeeman matrix element to the electric dipole matrix element in an isolated atomic orbital of radius a_0 scales as:

$$\frac{|\langle a|\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_{\text{opt}}|b\rangle|}{|\langle a|e\hat{\mathbf{r}}\cdot\mathbf{E}_{\text{opt}}|b\rangle|} \sim \frac{\mu_B B_0}{ea_0E_0} = \frac{\hbar}{2m_eca_0} \approx \frac{\alpha}{2} \sim 10^{-3}, \tag{2}$$

with $\alpha \approx 1/137$. This suppression has long justified neglecting $\mathbf{B}_{\text{opt}}$ in optical-frequency condensed matter physics.

In multi-band systems with strong spin-orbit coupling, the kinematic band velocity operator $\hat{\mathbf{v}}(\mathbf{k}) = \frac{1}{\hbar}\nabla_{\mathbf{k}}\hat{\mathscr{H}}_0$ contains both a scalar kinetic component $\mathbf{v}_0(\mathbf{k}) = \frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_0$ and a spin-dependent component $\hat{\mathbf{v}}_{\text{spin}}(\mathbf{k}) = \frac{1}{\hbar}\sum_a(\nabla_{\mathbf{k}}d_a)\sigma_a$. Consequently, the atomic scaling must be replaced by two fundamental dimensionless ratios defined directly through the local field amplitudes:

$$\eta_Z^{(0)} \equiv \frac{\mu_B B_0}{e v_0 A_0}, \tag{3}$$

$$\eta_Z^{(\text{spin})} \equiv \frac{\mu_B B_0}{e v_{\text{spin}} A_0} = \frac{\hbar\mu_B B_0}{e\|\nabla_{\mathbf{k}}\mathbf{d}\|A_0}, \tag{4}$$

where $v_0 \equiv \frac{1}{\hbar}\|\nabla_{\mathbf{k}}\varepsilon_0\|$ and $v_{\text{spin}} \equiv \frac{1}{\hbar}\|\nabla_{\mathbf{k}}\mathbf{d}\|$. In the asymptotic limit of a transverse vacuum plane wave, the relation $B_0 \simeq (\Omega/c)A_0$ yields $\eta_Z^{(0)} \simeq \frac{\hbar\Omega}{2m_e c v_0}$; however, Eqs. (3)–(4) remain valid in dielectrics, polaritonic media, and structured near-fields where the local ratio B_0/A_0 departs substantially from Ω/c.

These parameters delineate distinct physical consequences:

1. **Quenching of Scalar Kinetic Dispersion ($\eta_Z^{(0)} \gg 1$):** In flat-band systems where scalar dispersion is quenched ($v_0 \to 0$), the ratio $\eta_Z^{(0)}$ diverges. This suppresses the conventional intraband kinetic drive. However, this does not imply that all electric coupling vanishes, as the spin-dependent interaction $e\hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}$ remains active.
2. **Relative Spin-Channel Coupling ($\eta_Z^{(\text{spin})}$):** The parameter $\eta_Z^{(\text{spin})}$ sets the baseline scale for the optical Zeeman energy relative to the spin-dependent electric interaction. Whether the Zeeman interaction contributes significantly to non-equilibrium band reconstruction depends on how this ratio couples to the momentum-space geometry of the band structure.

Furthermore, in the presence of macroscopic magnetic order $\mathbf{M}(\mathbf{r},t)$, the optical Zeeman torque density $\boldsymbol{\tau}_Z = -\gamma\mu_0(\mathbf{M}\times\mathbf{H}_{\text{opt}})$ couples directly to the collective order parameter, bypassing virtual electronic interband transitions entirely.

2 Separation of Interaction Channels

To distinguish high-frequency magnetic torque from axionic field theories, we classify light-matter interactions into two distinct physical channels:

- **Channel I: Optomagnetic Zeeman-Floquet Driving.** Governed by the microscopic Zeeman interaction $\hat{\mathscr{H}}_Z = -\mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_{\text{opt}}(t)$. This channel operates in standard transverse plane waves where $\mathbf{E}_{\text{opt}} \perp \mathbf{B}_{\text{opt}}$ ($\mathbf{E}_{\text{opt}}\cdot\mathbf{B}_{\text{opt}} \equiv 0$). It reconfigures spin textures and induces secular effective fields via high-frequency operator commutators.
- **Channel II: Pseudoscalar Field Coupling.** Governed by the electromagnetic invariant $\mathscr{I}_2 = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \mathbf{E}_{\text{opt}}\cdot\mathbf{B}_{\text{opt}}$. This channel requires breaking transverse plane-wave symmetry (e.g., tightly focused non-paraxial beams with numerical aperture NA $\gtrsim 0.8$, non-collinear multi-beam interference, or chiral near-fields). It couples directly to topological axion angles $\theta(\mathbf{r},t)$ in three dimensions.

3 Channel I: Microscopic Derivation and Operator Competition Ratio

Consider a two-dimensional Bloch electron in a spin-orbit-coupled band described by:

$$\hat{\mathscr{H}}_0(\mathbf{k}) = \varepsilon_0(\mathbf{k})\mathbb{I} + \mathbf{d}(\mathbf{k})\cdot\boldsymbol{\sigma}, \tag{5}$$

where $\mathbf{d}(\mathbf{k}) = (d_x(\mathbf{k}), d_y(\mathbf{k}), d_z(\mathbf{k}))$ is the momentum-space spin-orbit field. Under monochromatic electromagnetic driving with vector potential $\mathbf{A}(t) = \text{Re}[\mathbf{A}_0 e^{-i\Omega t}]$ and optical magnetic field $\mathbf{B}_{\text{opt}}(t) = \text{Re}[\mathbf{B}_0 e^{-i\Omega t}]$, minimal coupling yields:

$$\hat{\mathscr{H}}(t) = \hat{\mathscr{H}}_0\left(\mathbf{k} + \frac{e}{\hbar}\mathbf{A}(t)\right) - \mu_B\boldsymbol{\sigma}\cdot\mathbf{B}_{\text{opt}}(t). \tag{6}$$

Expanding to first order in the field amplitudes gives:

$$\hat{\mathscr{H}}(t) \approx \hat{\mathscr{H}}_0(\mathbf{k}) + e\hat{\mathbf{v}}(\mathbf{k}) \cdot \mathbf{A}(t) - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_{\text{opt}}(t), \tag{7}$$

where the kinematic band velocity operator decomposes as:

$$\hat{\mathbf{v}}(\mathbf{k}) = \mathbf{v}_0(\mathbf{k})\mathbb{I} + \hat{\mathbf{v}}_{\text{spin}}(\mathbf{k}), \quad \mathbf{v}_0 = \frac{1}{\hbar}\nabla_{\mathbf{k}}\varepsilon_0, \quad \hat{\mathbf{v}}_{\text{spin}} = \frac{1}{\hbar}\sum_{a=1}^{3}(\nabla_{\mathbf{k}} d_a)\sigma_a. \tag{8}$$

3.1 Magnus Expansion Validity Regime

The stroboscopic Floquet Hamiltonian $\hat{\mathscr{H}}_{\text{F}}$ is determined by the Magnus expansion:

$$\hat{\mathscr{H}}_{\text{F}} = \hat{\mathscr{H}}_0 + \hat{\mathscr{H}}^{(1)}_{\text{electric}} + \hat{\mathscr{H}}^{(1)}_{\text{mixed}} + \hat{\mathscr{H}}^{(1)}_{\text{Zeeman}} + \mathscr{O}\left(\frac{1}{\Omega^2}\right), \tag{9}$$

where the first-order corrections originate from the Magnus commutator $\frac{1}{\hbar\Omega}[\hat{\mathscr{H}}_1, \hat{\mathscr{H}}_{-1}]$, with harmonic components $\hat{\mathscr{H}}_1 = \frac{1}{2}(e\hat{\mathbf{v}} \cdot \mathbf{A}_0 - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_0)$ and $\hat{\mathscr{H}}_{-1} = \hat{\mathscr{H}}_1^\dagger = \frac{1}{2}(e\hat{\mathbf{v}} \cdot \mathbf{A}_0^* - \mu_B \boldsymbol{\sigma} \cdot \mathbf{B}_0^*)$. This high-frequency expansion is strictly asymptotic and controlled under three simultaneous physical conditions:

1. **High-Frequency Carrier Separation:** $\hbar\Omega \gg W$, where W is the unperturbed electronic bandwidth.
2. **Perturbative Driving Amplitude:** $\|\hat{\mathscr{H}}_{\pm 1}\| \ll \hbar\Omega$, ensuring the drive does not induce uncontrolled multi-photon resonance crossings.
3. **Gauge Linearization Bound:** $\frac{e|\mathbf{A}_0|}{\hbar} \ll \Lambda_k$, where Λ_k represents the characteristic momentum-space variation scale of $\nabla_{\mathbf{k}}\mathbf{d}(\mathbf{k})$ (e.g., the moiré Brillouin zone dimension).

3.2 Evaluation of the Mixed Optomagnetic Commutator

The mixed optomagnetic operator appearing in Eq. (9) is:

$$\hat{\mathscr{H}}^{(1)}_{\text{mixed}} = -\frac{e\mu_B}{4\hbar\Omega}\left([\hat{\mathbf{v}} \cdot \mathbf{A}_0, \boldsymbol{\sigma} \cdot \mathbf{B}_0^*] + [\boldsymbol{\sigma} \cdot \mathbf{B}_0, \hat{\mathbf{v}} \cdot \mathbf{A}_0^*]\right). \tag{10}$$

Because $[\mathbb{I}, \sigma_b] = 0$, the scalar velocity $\mathbf{v}_0$ drops out identically from both commutators. Expanding the scalar products $\boldsymbol{\sigma} \cdot \mathbf{B}_0 = \sum_{b=1}^{3} B_{0,b}\sigma_b$ and $\hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}_0 = \frac{1}{\hbar}\sum_{i=1}^{2}\sum_{a=1}^{3}(\partial_{k_i} d_a)A_{0,i}\sigma_a$, we evaluate each bracket:

$$[\hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}_0, \boldsymbol{\sigma} \cdot \mathbf{B}_0^*] = \frac{1}{\hbar}\sum_{i=1}^{2}\sum_{a,b=1}^{3}(\partial_{k_i} d_a)A_{0,i}B^*_{0,b}[\sigma_a, \sigma_b] = \frac{2i}{\hbar}\sum_{i=1}^{2}\sum_{a,b,c=1}^{3}\varepsilon_{abc}(\partial_{k_i} d_a)A_{0,i}B^*_{0,b}\sigma_c, \tag{11}$$

$$[\boldsymbol{\sigma} \cdot \mathbf{B}_0, \hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}_0^*] = \frac{1}{\hbar}\sum_{i=1}^{2}\sum_{a,b=1}^{3}(\partial_{k_i} d_a)B_{0,b}A^*_{0,i}[\sigma_b, \sigma_a] = -\frac{2i}{\hbar}\sum_{i=1}^{2}\sum_{a,b,c=1}^{3}\varepsilon_{abc}(\partial_{k_i} d_a)A^*_{0,i}B_{0,b}\sigma_c. \tag{12}$$

Summing these two contributions inside Eq. (10) yields:

$$\begin{aligned}[\hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}_0, \boldsymbol{\sigma} \cdot \mathbf{B}_0^*] + [\boldsymbol{\sigma} \cdot \mathbf{B}_0, \hat{\mathbf{v}}_{\text{spin}} \cdot \mathbf{A}_0^*] &= \frac{2i}{\hbar}\sum_{i=1}^{2}\sum_{a,b,c=1}^{3}\varepsilon_{abc}(\partial_{k_i} d_a)\left(A_{0,i}B^*_{0,b} - A^*_{0,i}B_{0,b}\right)\sigma_c \\ &= -\frac{4}{\hbar}\sum_{i=1}^{2}\sum_{a,b,c=1}^{3}\varepsilon_{abc}(\partial_{k_i} d_a)\,\text{Im}\left[A_{0,i}B^*_{0,b}\right]\sigma_c.\end{aligned} \tag{13}$$

Substituting this back into Eq. (10) produces the effective secular Zeeman interaction:

$$\hat{\mathscr{H}}^{(1)}_{\text{mixed}} = \mathbf{h}_{\text{ind}}(\mathbf{k}) \cdot \boldsymbol{\sigma}, \quad \text{with} \quad \mathbf{h}^c_{\text{ind}}(\mathbf{k}) \equiv \frac{e\mu_B}{\hbar^2\Omega}\sum_{i=1}^{2}\sum_{a,b=1}^{3}\varepsilon_{abc}(\partial_{k_i} d_a)\,\text{Im}\left[A_{0,i}B^*_{0,b}\right]. \tag{14}$$

Because $\text{Im}[A_{0,i}B^*_{0,b}]$ and $\partial_{k_i} d_a$ are real and $\sigma_c^\dagger = \sigma_c$, the operator $\mathbf{h}_{\text{ind}}(\mathbf{k}) \cdot \boldsymbol{\sigma}$ is manifestly Hermitian.

3.3 The Operator Competition Ratio $\mathscr{R}_{\mathbf{mix}}$

To evaluate whether $\hat{\mathscr{H}}^{(1)}_{\text{mixed}}$ is physically significant, it must be compared to the pure electric Floquet term $\hat{\mathscr{H}}^{(1)}_{\text{electric}} = \frac{e^2}{4\hbar\Omega}[\hat{\mathbf{v}}\cdot\mathbf{A}_0,\, \hat{\mathbf{v}}\cdot\mathbf{A}_0^*]$. The commutator of the velocity operator evaluates to:

$$[\hat{\mathbf{v}}\cdot\mathbf{A}_0,\, \hat{\mathbf{v}}\cdot\mathbf{A}_0^*] = [\hat{\mathbf{v}}_{\text{spin}}\cdot\mathbf{A}_0,\, \hat{\mathbf{v}}_{\text{spin}}\cdot\mathbf{A}_0^*] = \frac{4i}{\hbar^2}\sum_{i,j=1}^{2}\sum_{a,b,c=1}^{3}\varepsilon_{abc}(\partial_{k_i}d_a)(\partial_{k_j}d_b)\,\mathrm{Im}[A_{0,i}A_{0,j}^*]\sigma_c. \tag{15}$$

We define the dimensionless operator competition ratio:

$$\mathscr{R}_{\text{mix}}(\mathbf{k}) \equiv \frac{\|\hat{\mathscr{H}}^{(1)}_{\text{mixed}}(\mathbf{k})\|}{\|\hat{\mathscr{H}}^{(1)}_{\text{electric}}(\mathbf{k})\|} = \eta_Z^{(\text{spin})}\cdot\frac{\left\|\sum_{i=1}^{2}\sum_{a,b=1}^{3}\varepsilon_{abc}\left(\frac{\partial_{k_i}d_a}{\|\nabla_{\mathbf{k}}\mathbf{d}\|}\right)\mathrm{Im}\left[\frac{A_{0,i}B_{0,b}^*}{A_0B_0}\right]\right\|}{\left\|\sum_{i,j=1}^{2}\sum_{a,b=1}^{3}\varepsilon_{abc}\left(\frac{\partial_{k_i}d_a}{\|\nabla_{\mathbf{k}}\mathbf{d}\|}\right)\left(\frac{\partial_{k_j}d_b}{\|\nabla_{\mathbf{k}}\mathbf{d}\|}\right)\mathrm{Im}\left[\frac{A_{0,i}A_{0,j}^*}{A_0^2}\right]\right\|}. \tag{16}$$

Equation (16) demonstrates that the physical relevance of the mixed optomagnetic interaction is not determined solely by $\eta_Z^{(\text{spin})}$, but depends on the geometric projection of the driving polarization onto the spin-orbit texture. Crucially, in driving geometries where the electric commutator vanishes identically (such as linearly polarized drives where $\mathrm{Im}[A_{0,i}A_{0,j}^*] = 0$), the mixed term provides the leading non-vanishing Floquet correction whenever elliptical or non-paraxial magnetic components maintain $\mathrm{Im}[A_{0,i}B_{0,b}^*] \neq 0$.

4 Nonequilibrium Topology in the Quasi-Floquet Regime

When the optical drive is applied as a pulse with finite envelope $\lambda(t)$, discrete time-translation symmetry is broken. However, in the adiabatic quasi-Floquet regime, two conditions must be satisfied:

1. **Carrier-Envelope Separation:**
$$\left|\frac{1}{\lambda(t)}\frac{d\lambda}{dt}\right| \ll \Omega. \tag{17}$$
2. **Spectral Gap Adiabaticity:**
$$\max_{\alpha\neq\beta}\frac{\hbar\left|\langle u_\alpha(t)|\partial_t|u_\beta(t)\rangle\right|}{|\varepsilon_\alpha(t)-\varepsilon_\beta(t)|^2} \ll 1, \tag{18}$$
where $|u_\alpha(t)\rangle$ and $\varepsilon_\alpha(t)$ denote the instantaneous Floquet eigenstates and quasienergies parameterized by $\lambda(t)$.

Under these conditions, the instantaneous (frozen-drive) Floquet spectrum is well-defined on the circle $\mathbb{S}^1 \equiv [-\hbar\Omega/2, \hbar\Omega/2)$.

The non-equilibrium topological classification is governed by two independent winding invariants, W_0 and W_π, evaluated for the 0-gap ($\varepsilon \sim 0$) and the π-gap ($\varepsilon \sim \pm\hbar\Omega/2$). In a two-band system, the Chern number of the lower Floquet band satisfies $\mathscr{C} = W_0 - W_\pi$.

Proposition 1 (Quasi-Floquet Gap Invariance Criteria)**.** *During a smooth pulse envelope $\lambda(t)$ satisfying quasi-Floquet adiabaticity, the invariants W_0 and W_π remain strictly constant unless the instantaneous spectrum closes the respective gap:*

$$\Delta W_0 \neq 0 \implies \exists(\mathbf{k}_c, t_c) \quad \textit{such that} \quad \varepsilon_\alpha(\mathbf{k}_c,\lambda(t_c)) = \varepsilon_\beta(\mathbf{k}_c,\lambda(t_c)) \equiv 0 \pmod{\hbar\Omega}, \tag{19}$$

$$\Delta W_\pi \neq 0 \implies \exists(\mathbf{k}_c, t_c) \quad \textit{such that} \quad \varepsilon_\alpha(\mathbf{k}_c,\lambda(t_c)) = \varepsilon_\beta(\mathbf{k}_c,\lambda(t_c)) \equiv \pm\frac{\hbar\Omega}{2} \pmod{\hbar\Omega}. \tag{20}$$

4.1 Theoretical Dichotomy for Moiré Heterostructures

Recent experiments have demonstrated deterministic optical reversal of spin-valley order and Chern ferromagnetism in twisted $MoTe_2$, observed under ultrafast pulsed excitation (Huber et al., 2026) and continuous-wave illumination (Cai et al., 2026). This phenomenology admits two distinct microscopic interpretations:

1. **Coherent Floquet Gap Inversion:** The periodic drive dynamically renormalizes the moiré bands via Eq. (14), driving the system through an explicit quasienergy gap closing [Eqs. (19)–(20)] on sub-cycle time scales without significant electronic entropy production.

2. **Exciton-Polaron-Mediated Switching:** The optical field resonantly excites valley-polarized exciton-polarons. The resulting non-equilibrium exchange field acts as a thermodynamic torque that nucleates domain reversal across a first-order energetic landscape.

The experimental observation of continuous-wave switching is consistent with incoherent exciton-mediated mechanisms playing a primary role in steady-state regimes. Determining whether coherent Floquet gap closing occurs during ultrashort pulsed excitation requires time-resolved angle-resolved photoemission spectroscopy (tr-ARPES) to probe whether quasienergy gaps close dynamically during the optical pulse.

5 Channel II: Regularized Lattice Chern–Simons Action

We now examine Channel II. To avoid the regularization ambiguities of continuum models, we define the two-dimensional electronic system on a discrete spatial lattice.

5.1 Lattice-Regularized Adiabatic Response

Consider a generic two-band lattice Hamiltonian defined over the compact two-torus of the first Brillouin zone ($\mathbf{k} \in T^2$):

$$\hat{\mathscr{H}}_{\text{latt}}(\mathbf{k}) = \mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma} = d_x(\mathbf{k})\sigma_x + d_y(\mathbf{k})\sigma_y + d_z(\mathbf{k}; n_z)\sigma_z, \tag{21}$$

where $d_z(\mathbf{k}; n_z) = m(\mathbf{k}) + \Delta_{\text{ex}} n_z(\mathbf{r}, t)$, with $n_z = \mathbf{M} \cdot \hat{\mathbf{z}}/M_s$. For a gapped insulator ($|\mathbf{d}(\mathbf{k})| > 0$ everywhere on T^2), the unit vector $\hat{\mathbf{d}}(\mathbf{k}) = \mathbf{d}(\mathbf{k})/|\mathbf{d}(\mathbf{k})|$ defines a smooth mapping $T^2 \to S^2$.

Integrating out the valence fermions in the presence of a slowly varying gauge field A_μ yields the effective action $\mathscr{S}_{\text{eff}}[A] = -i \operatorname{Tr} \ln[i\partial_t - \hat{\mathscr{H}}_{\text{latt}}(\mathbf{k} + e\mathbf{A})]$. In the adiabatic limit, the parity-odd component yields the strictly integer-quantized Chern–Simons response:

$$\mathscr{S}_{\text{CS}}[A, n_z] = \frac{\sigma_{xy}[n_z]}{2} \int d^3x \, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{22}$$

where the Hall conductivity is determined by the Pontryagin index of $\hat{\mathbf{d}}$:

$$\sigma_{xy}[n_z] = \frac{e^2}{h} \mathscr{C}[n_z], \quad \text{with} \quad \mathscr{C}[n_z] = \frac{1}{4\pi} \int_{\text{BZ}} d^2k \, \hat{\mathbf{d}} \cdot \left(\partial_{k_x} \hat{\mathbf{d}} \times \partial_{k_y} \hat{\mathbf{d}} \right) \in \mathbb{Z}. \tag{23}$$

Condition 1 (Lattice Gap Closing and Topological Transition)**.** *A continuous parameter trajectory $n_z(t)$ interpolating from $+1$ to -1 can induce a discrete change in the topological Chern invariant ($\Delta\mathscr{C} \neq 0$) only if the bulk spectral gap closes during the interpolation. For an isolated, linear (non-degenerate) nodal crossing at momentum $\mathbf{k}_0 \in T^2$ where:*

$$d_x(\mathbf{k}_0) = 0, \quad d_y(\mathbf{k}_0) = 0, \quad \textit{and} \quad sgn[d_z(\mathbf{k}_0; +1)] = -sgn[d_z(\mathbf{k}_0; -1)], \tag{24}$$

the zero-crossing $d_z(\mathbf{k}_0; n_z(t_0)) = 0$ inverts the local mass sign. This crossing alters the global Chern integer $\mathscr{C}$ by ± 1, provided the sum of topological charges over all simultaneous gap-closing points in the Brillouin zone is non-zero.

5.2 Formulation of the Non-Equilibrium Keldysh-Floquet Extension

When the driving frequency Ω is comparable to the band gap ($\hbar\Omega \sim |\mathbf{d}|$), the adiabatic derivative expansion is invalid. The full non-equilibrium effective action must be evaluated on the Schwinger–Keldysh closed-time contour $\mathscr{C}$:

$$\mathscr{S}_{\text{eff}}[A,n_z] = -i\,\mathrm{Tr}_{\mathscr{C}} \ln\left[G_0^{-1} - \hat{\Sigma}[A,n_z]\right], \tag{25}$$

where G_0 is the unperturbed contour Green's function. Projecting into the Floquet steady state yields the non-local current response:

$$j^{\mu}(\mathbf{q},\omega) = \sum_{m\in\mathbb{Z}} \Pi^{\mu\nu}_{(m)}(\mathbf{q},\omega) A_{\nu}(\mathbf{q},\omega - m\Omega). \tag{26}$$

A local Chern–Simons description is recovered only in the off-resonant limit ($\hbar\Omega \gg W$), where the $m \neq 0$ sidebands can be integrated out perturbatively to yield the renormalized Hall conductivity of the Floquet state.

5.3 The $(3+1)$D Bulk Axion Action and Structured Light Fields

In three-dimensional magnetic topological insulators (e.g., stoichiometric $MnBi_2Te_4$), the functional determinant generates the bulk axionic action:

$$\mathscr{S}_{\theta} = \frac{e^2}{32\pi^2} \int d^4x\, \theta[\mathbf{n}(\mathbf{r},t)]\, \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} = \frac{e^2}{4\pi^2} \int d^4x\, \theta[\mathbf{n}(\mathbf{r},t)] \left(\mathbf{E}_{\text{opt}} \cdot \mathbf{B}_{\text{opt}}\right). \tag{27}$$

Because an ideal transverse plane wave in vacuum has $\mathbf{E}_{\text{opt}} \cdot \mathbf{B}_{\text{opt}} \equiv 0$, Channel II dynamics require structured light fields:

1. **Non-Paraxial Focal Fields:** For tightly focused beams (NA $\gtrsim 0.8$), longitudinal fields E_z and B_z can be engineered with spatial overlap, yielding $\int_V (\mathbf{E}_{\text{opt}} \cdot \mathbf{B}_{\text{opt}}) d^3r \neq 0$.
2. **Multi-Beam Interference:** Superposing two coherent, non-collinear waves ($\mathbf{k}_1 \nparallel \mathbf{k}_2$) with orthogonal polarizations generates a non-vanishing local pseudoscalar density $\mathbf{E}_{\text{opt}} \cdot \mathbf{B}_{\text{opt}} \propto \sin(\theta_{12}) \sin(\Delta\phi)$.

6 Conclusion

The 2025–2026 experimental demonstrations of optical magnetic torque and non-thermal Chern domain switching provide a rigorous foundation for non-equilibrium materials design when evaluated within a complete theoretical framework:

1. Direct optical Zeeman driving generates a manifestly Hermitian secular torque in the effective Floquet Hamiltonian at order $\mathscr{O}(\Omega^{-1})$ via the spin-dependent velocity $\hat{\mathbf{v}}_{\text{spin}} = \frac{1}{\hbar}(\nabla_{\mathbf{k}}\mathbf{d}) \cdot \boldsymbol{\sigma}$.
2. The competition between mixed optomagnetic and pure electric driving is governed by the operator ratio $\mathscr{R}_{\text{mix}}(\mathbf{k})$, demonstrating that quenching scalar kinetic velocity ($v_0 \to 0$) isolates the spin-dependent interaction channel where polarization geometry dictates the dominant Floquet term.
3. In the quasi-Floquet adiabatic regime, coherent topological state changes strictly require closing either the 0- or π-quasienergy gaps, providing a clear spectroscopic signature to distinguish coherent Floquet driving from exciton-polaron-mediated switching.
4. Pseudoscalar axion electrodynamics ($\mathbf{E}_{\text{opt}} \cdot \mathbf{B}_{\text{opt}} \neq 0$) does not govern standard plane-wave optomagnetism, but represents a distinct interaction channel accessible via structured and non-paraxial optical fields.

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