

# A Theoretical Framework for Radion-Supported Gravitating Solitons: Towards a Geometric Interpretation of Confined Gauge-Field Mass

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## Author's Note and Open Invitation

*As the author has transitioned into retirement, the extensive numerical computation and stability analysis required to complete this research program fall beyond current personal scope [27]. This manuscript is therefore presented as a rigorous, open mathematical framework defining the Radion-Supported Gravitating Soliton Existence Conjecture (RSGSEC). The explicit numerical construction of these solitons, the calculation of their mass spectra, and the evaluation of their linear quasinormal perturbation modes are offered as an open challenge to the theoretical physics and numerical relativity communities [27].*

## Abstract

We investigate whether localized, finite-energy solutions of an Einstein-Yang-Mills-Higgs (EYMh) system coupled to a dynamically stabilized radion field admit an interpretation in which four-dimensional mass emerges from confined gauge-field flux and geometry—a phenomenon that may arise if regular, dynamically stable solutions exist. By performing a dimensional reduction of a  $(4 + n)$ -dimensional bulk action over a compact internal Einstein manifold  $\mathcal{K}^n$ , we extract a four-dimensional effective field theory (EFT) within a zero-mode Kaluza-Klein truncation where the canonical radion field dynamically couples to the Yang-Mills and Higgs kinetic sectors. Within this low-energy EFT truncation, internal curvature and quantized harmonic flux generate a minimal classical effective potential  $V_{eff}(\phi)$  with a positive-mass minimum, ensuring radion stabilization for internal dimensions  $n \geq 2$ . This structure formulates the Radion-Supported Gravitating Soliton Existence Conjecture (RSGSEC). We derive all exponential couplings from a unified master Weyl formula, obtain the five coupled radial field equations with exact canonical dimensional uniformity, prove covariant stress-energy conservation, establish the total ADM energy functional and Derrick-Pohozaev virial identity, demonstrate analytical robustness against Derrick's and Bekenstein's no-go theorems, verify exact reductions to classical EYMh, 't Hooft-Polyakov, and Bartnik-McKinnon limits, derive the second-variation linear quasinormal perturbation matrix (including the explicit closed-form radion component  $\mathbf{U}_{eff,\phi\phi}(r)$ ), formalize the boxed RSGSEC statement, and outline one possible functional-analysis strategy alongside numerical continuation programs required to test whether these equations admit regular, dynamically stable self-gravitating defects.

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## 1 Introduction and Literature Review

Standard classical and quantum field theories operate under a paradigm where matter fields act as sources dictating the curvature of a spacetime arena. However, the hypothesis that stable, localized matter states might emerge dynamically from continuous field configurations—or pure geometry—possesses a profound theoretical history. Foundational concepts such as Wheeler’s geons [1], ’t Hooft-Polyakov magnetic monopoles [2, 3], Skyrmions [4], and boson stars [5–7]

demonstrated that non-trivial topology, conserved Noether charges, and non-Abelian gauge self-interactions can stabilize particle-like field structures against dispersion.

In the realm of general relativity without Higgs scalars, Bartnik and McKinnon discovered that pure Einstein-Yang-Mills (EYM) theory admits a discrete family of regular, static, self-gravitating solitons [8, 9]. When spontaneous symmetry breaking via an adjoint Higgs scalar is introduced, the system expands into the Einstein-Yang-Mills-Higgs (EYMH) framework, admitting gravitating magnetic monopoles and non-Abelian black holes with hair [10–14].

Concurrently, higher-dimensional unification frameworks such as Kaluza-Klein theory [15, 16], Freund-Rubin compactifications [17], and Randall-Sundrum braneworld models [18] utilize compact extra dimensions together with dynamic moduli stabilization—such as the Goldberger-Wise mechanism [19] and flux-induced string compactification schemes [20, 21]—to explain fundamental energy hierarchies. Foundational investigations by Gross, Perry, Sorkin, and Breitenlohner et al. [22–24] demonstrated how magnetic monopoles embed within Kaluza-Klein geometries. Comprehensive reviews of Kaluza-Klein compactifications and effective field theory truncations can be found in Duff, Nilsson, and Pope [25] and Bailin and Love [26].

| Soliton / Defect Family                      | Higgs Field | Gravity | Extra Dimensions | Dynamic Radion         |
|----------------------------------------------|-------------|---------|------------------|------------------------|
| 't Hooft-Polyakov Monopole [2, 3]            | ✓           | ×       | ×                | ×                      |
| Bartnik-McKinnon Soliton [8]                 | ×           | ✓       | ×                | ×                      |
| Boson Stars / Nontopological Solitons [5, 7] | Scalar Only | ✓       | ×                | ×                      |
| EYMH Gravitating Monopoles [9, 10]           | ✓           | ✓       | ×                | ×                      |
| Kaluza-Klein Monopoles [22, 23]              | ×           | ✓       | ✓                | ×                      |
| <b>RSGSEC (Present Work)</b>                 | ✓           | ✓       | ✓                | × (Static/Global)<br>✓ |

Table 1: Comparison of the Radion-Supported Gravitating Soliton Existence Conjecture (RSGSEC) framework with established soliton and topological defect families.

As illustrated in Table 1, our proposal distinguishes itself from established soliton families by explicitly coupling a dynamically stabilized scalar radion field to localized non-Abelian gauge flux within the defect core. Rather than assuming the compactification volume remains a rigid global constant across spacetime, the internal geometry dynamically responds to local Yang-Mills and Higgs energy densities, shifting from its asymptotic vacuum expectation value to a locally excited configuration inside the soliton core.

## 2 Dimensional Reduction, Truncation Scope, and Effective Action

### 2.1 The $(4+n)$ -Dimensional Bulk Action and Warped Metric Decomposition

To formalize the derivation of the effective four-dimensional theory, we consider a  $(4+n)$ -dimensional bulk manifold  $\mathcal{M}^{4+n}$  with signature  $(-, +, +, \dots, +)$ , coupled to an  $SU(2)$  gauge field  $A_M^a$  and a Higgs scalar sector  $\Phi^a$ . The total bulk action is given by:

$$S = \int d^{4+n}X \sqrt{-G} \left[ \frac{M_*^{2+n}}{2} \mathcal{R}^{(4+n)} - \frac{1}{4} F_{MN}^a F^{MN,a} - \frac{1}{2} (D_M \Phi)^a (D^M \Phi)^a - U_{bulk}(\Phi) - \Lambda_{bulk} \right], \quad (1)$$

where  $M_*$  is the fundamental  $(4+n)$ -dimensional Planck scale,  $G_{MN}$  is the bulk metric with determinant  $G$ , and  $F_{MN}^a = \partial_M A_N^a - \partial_N A_M^a + e\epsilon^{abc} A_M^b A_N^c$  is the non-Abelian field strength tensor defined with canonical gauge coupling  $e$ .

We assume the bulk manifold decomposes topologically as  $\mathcal{M}^{4+n} = \mathcal{M}^4 \times \kappa^n$ , where  $\kappa^n$  is a smooth, compact,  $n$ -dimensional internal Einstein manifold with baseline metric  $\gamma_{ab}(y)$  and constant positive scalar curvature  $\mathcal{R}_\kappa > 0$ . We adopt the factorizable Kaluza-Klein warped

metric ansatz:

$$ds_{4+n}^2 = G_{MN} dX^M dX^N = g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\sigma(x)} \gamma_{ab}(y) dy^a dy^b, \quad (2)$$

where  $\mu, \nu \in \{0, 1, 2, 3\}$  index external four-dimensional spacetime,  $a, b \in \{4, \dots, 3+n\}$  index the internal manifold  $\kappa^n$ , and  $\sigma(x)$  is the dimensionless breathing mode (modulus) controlling the local volume of the extra dimensions. Evaluating the  $(4+n)$ -dimensional Christoffel symbols for the warped metric (2) yields the exact dimensional reduction decomposition of the scalar curvature [17, 25, 26]:

$$\mathcal{R}^{(4+n)} = \mathcal{R}^{(4)}(g) + e^{-2\sigma} \mathcal{R}_\kappa - 2n \square_g \sigma - n(n+1) g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma. \quad (3)$$

## 2.2 Low-Energy Effective Field Theory Truncation Scope

We explicitly formalize the distinction between an exact *consistent truncation* and a *low-energy effective field theory (EFT) truncation*. A truncation is strictly consistent if all massive Kaluza-Klein excitation modes remain identically zero across all orders of the equations of motion. While exact consistent truncations exist for specific symmetric coset reductions (such as spheres  $S^n$  with group-invariant ansatzes [25]), for general internal Einstein spaces  $\kappa^n$ , our single-radion zero-mode ansatz (2) constitutes a low-energy EFT truncation.

To guarantee that massive Kaluza-Klein modes do not become excited, our framework explicitly imposes the **Kaluza-Klein Decoupling Condition**: the characteristic energy scale of the soliton core ( $E_{\text{core}} \sim v$ ) and its radial curvature gradients must remain strictly below the lowest massive Kaluza-Klein excitation threshold ( $M_{KK} \sim R_0^{-1}$ ), namely:

$$v \ll M_{KK} \sim \frac{1}{R_0}. \quad (4)$$

Under this condition, source terms generating higher Kaluza-Klein tower modes are suppressed by powers of  $(vR_0)^2 \ll 1$ , ensuring that massive modes decouple and the zero-mode EFT truncation remains mathematically self-consistent.

## 2.3 Master Conformal Weyl Scaling Formula and Four-Dimensional Action

Integrating over the internal volume  $V_K = \int d^n y \sqrt{\gamma}$  and applying the conformal Weyl transformation from the Jordan frame ( $g_{\mu\nu}$ ) to the canonical four-dimensional Einstein frame ( $\tilde{g}_{\mu\nu}$ ):

$$g_{\mu\nu}(x) = e^{-n\sigma(x)} \tilde{g}_{\mu\nu}(x) \implies g^{\mu\nu}(x) = e^{n\sigma(x)} \tilde{g}^{\mu\nu}(x), \quad \sqrt{-g} = e^{-2n\sigma} \sqrt{-\tilde{g}}, \quad (5)$$

the complete step-by-step derivation of the four-dimensional EFT action and canonical radion normalization  $\sigma(x) = \beta \frac{\phi(x)}{M_{Pl}}$  (where  $\beta \equiv \sqrt{\frac{2}{n(n+2)}}$ ) is presented in Appendix A.

To ensure algebraic elegance and eliminate convention ambiguity across all sectors, we derive all exponential couplings from a **Master Weyl Coupling Formula**. Let  $w_i$  be the conformal Weyl scaling weight of any Lagrangian term  $\mathcal{L}_i^{(J)}$  upon transition from the Jordan frame to the canonical four-dimensional Einstein frame under Eq. (5):

$$\sqrt{-g} \mathcal{L}_i^{(J)} = \sqrt{-\tilde{g}} e^{w_i \sigma} \mathcal{L}_i^{(E)}. \quad (6)$$

Substituting  $\sigma = \beta \frac{\phi}{M_{Pl}}$ , the effective coupling function for the  $i$ -th sector takes the universal master exponential form:

$$\mathcal{F}_i(\phi) = \exp\left(w_i \beta \frac{\phi}{M_{Pl}}\right) \equiv \exp\left(-c_i \frac{\phi}{M_{Pl}}\right), \quad \text{where } c_i \equiv -w_i \beta = -w_i \sqrt{\frac{2}{n(n+2)}}. \quad (7)$$

Evaluating the exact Weyl weights  $w_i$  across all sectors of the action yields:

- **Yang-Mills Gauge Sector:**  $w_{YM} = +n \implies c_1 = -n\beta = -\sqrt{\frac{2n}{n+2}}$ .
- **Geometric Internal Gauge Higgs Sector ( $\Phi_m^a$  where  $m \in \kappa^n$ ):**  $w_H = +(n-2) \implies c_2 = -(n-2)\beta = -\sqrt{\frac{2(n-2)^2}{n(n+2)}}$ . (If  $\Phi^a$  is instead a fundamental four-dimensional adjoint scalar in the bulk,  $w_H = +n \implies c_2 = c_1$ ).
- **Internal Curvature Potential Term ( $Ae^{-a\phi}$ ):**  $w_{\mathcal{R}} = -(n+2) \implies a = -w_{\mathcal{R}}\beta = (n+2)\beta = \sqrt{\frac{2(n+2)}{n}}$ .
- **Freund-Rubin Harmonic Flux Term ( $Be^{-b\phi}$ ):**  $w_{flux} = -3n \implies b = -w_{flux}\beta = 3n\beta = \frac{3n}{\sqrt{n(n+2)/2}}$ .

The resulting four-dimensional Einstein-frame action within our low-energy zero-mode EFT truncation is precisely:

$$S_{eff} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{M_{Pl}^2}{2} \tilde{\mathcal{R}} - \frac{1}{2} (\tilde{\partial}\phi)^2 - V_{eff}(\phi) - \frac{1}{4} f(\phi) F_{\mu\nu}^a F^{\mu\nu,a} - \frac{1}{2} y(\phi) (D_\mu \Phi)^a (D^\mu \Phi)^a - U(\Phi) \right], \quad (8)$$

where  $f(\phi) = \exp(-c_1\phi/M_{Pl})$ ,  $y(\phi) = \exp(-c_2\phi/M_{Pl})$ ,  $D_\mu \Phi^a = \partial_\mu \Phi^a + e\epsilon^{abc} A_\mu^b \Phi^c$  is the gauge-covariant derivative, and  $U(\Phi) = \frac{\lambda}{4} (\Phi^a \Phi^a - v^2)^2$  is the symmetry-breaking potential. Throughout the main body of this work, we analyze the generalized geometric representation defined by Eq. (7).

## 2.4 Minimal Classical Radion Effective Potential $V_{eff}(\phi)$

In the absence of stabilizing internal flux or quantum corrections, a compact extra dimension governed solely by internal curvature  $\kappa_\kappa$  tends toward either rapid collapse ( $\sigma \rightarrow -\infty$ ) or decompactification ( $\sigma \rightarrow +\infty$ ). To guarantee a dynamically stable compactification with a positive squared mass within our classical EFT truncation, we introduce a Freund-Rubin stabilizing harmonic  $q$ -form flux on  $\kappa^n$  ( $q = n$ ) with quantized flux number  $N_{flux} \in \mathbb{Z}$  over non-contractible  $q$ -cycles  $\Sigma_q \subset \kappa^n$ :  $\int_{\Sigma_q} F^{(q)} = 2\pi N_{flux}$  [17].

Applying our master formula (7) with weights  $w_{\mathcal{R}} = -(n+2)$  and  $w_{flux} = -3n$ , alongside a four-dimensional cosmological counterterm  $\Lambda_0$  to tune the asymptotic vacuum energy to zero, we obtain the **Minimal Classical Radion Effective Potential**:

$$V_{eff}(\phi) = B \exp\left(-b \frac{\phi}{M_{Pl}}\right) - A \exp\left(-a \frac{\phi}{M_{Pl}}\right) + \Lambda_0, \quad (9)$$

where  $A = \frac{M_{Pl}^2 \mathcal{R}_\kappa}{2}$  and  $B = \frac{N_{flux}^2}{2M_{Pl}^2 R_0^{2n}}$ . We emphasize that Eq. (9) represents *one minimal classical choice* capturing the primary repulsive-attractive interplay. Realistic compactifications can generate quantum Casimir energies, higher-derivative curvature invariants, or localized brane tensions that introduce sub-leading exponentials around this minimal classical baseline.

We verify the condition under which the repulsive flux term dominates at small radii ( $\phi \rightarrow -\infty$ ), requiring strictly  $b > a$ :

$$b > a \iff b^2 > a^2 \iff \frac{18n^2}{n(n+2)} > \frac{2(n+2)^2}{n(n+2)} \iff 9n^2 > (n+2)^2 \iff (2n+1)(n-1) > 0. \quad (10)$$

For any integer internal dimension  $n \geq 2$ ,  $(2n+1)(n-1) > 0$  holds strictly ( $b > a$ ), ensuring that the potential possesses a unique, globally stable vacuum minimum  $\phi_0$  at finite compactification

radius. For  $n = 1$ ,  $a = b = \sqrt{6}$ , leading to degenerate scaling; stabilizing  $n = 1$  requires higher-order Casimir quantum loop corrections outside our minimal classical EFT truncation. Setting  $\left. \frac{\partial V_{eff}}{\partial \phi} \right|_{\phi_0} = 0$ , the vacuum expectation value  $\phi_0$  and positive radion mass squared  $m_\phi^2$  are:

$$\frac{\phi_0}{M_{Pl}} = \frac{1}{b-a} \ln \left( \frac{bB}{aA} \right), \quad m_\phi^2 = \left. \frac{\partial^2 V_{eff}}{\partial \phi^2} \right|_{\phi_0} = \frac{a(b-a)A}{M_{Pl}^2} \exp \left( -a \frac{\phi_0}{M_{Pl}} \right) > 0. \quad (11)$$

### 3 Variational Derivation, Stress-Energy Conservation, and Spherically Symmetric Radial Equations

#### 3.1 Four-Dimensional Field Equations and Spherically Symmetric Ansatz

Varying Eq. (8) independently with respect to  $\tilde{g}^{\mu\nu}$ ,  $A_\mu^a$ ,  $\Phi^a$ , and  $\phi$  yields the coupled four-dimensional zero-mode field equations:

$$\tilde{G}_{\mu\nu} = \frac{1}{M_{Pl}^2} T_{\mu\nu}^{(total)}, \quad (12)$$

$$D_\mu [f(\phi) F^{\mu\nu,a}] = e y(\phi) \epsilon^{abc} \Phi^b (D^\nu \Phi)^c, \quad (13)$$

$$D_\mu [y(\phi) D^\mu \Phi^a] = \frac{\partial U}{\partial \Phi^a}, \quad (14)$$

$$\tilde{\square} \phi = \frac{\partial V_{eff}}{\partial \phi} + \frac{1}{4} \frac{\partial f}{\partial \phi} F_{\mu\nu}^a F^{\mu\nu,a} + \frac{1}{2} \frac{\partial y}{\partial \phi} (D_\mu \Phi)^a (D^\mu \Phi)^a. \quad (15)$$

We parameterize the static, spherically symmetric spacetime metric in Schwarzschild-like coordinates as:

$$ds^2 = -N(r)S(r)^2 dt^2 + N(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (16)$$

where  $N(r) \equiv 1 - \frac{2Gm(r)}{r}$  defines the enclosed mass function  $m(r)$  with  $G = (8\pi M_{Pl}^2)^{-1}$ , and  $S(r)$  tracks the lapse deviation from pure Schwarzschild geometry. For the  $SU(2)$  gauge and Higgs fields, we adopt the standard static hedgehog ansatz:

$$A_0^a = 0, \quad A_i^a = \frac{1-K(r)}{er} \epsilon_{iaj} \frac{x^j}{r}, \quad \Phi^a = \frac{x^a}{r} H(r). \quad (17)$$

Substituting Eqs. (16)-(17) into  $T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$ , the total energy density  $\rho(r) = -T_0^0$  and radial pressure  $p_r(r) = T_r^r$  take explicit, closed forms:

$$\rho(r) = \frac{1}{2} N \phi'^2 + V_{eff}(\phi) + f(\phi) \left[ \frac{N K'^2}{e^2 r^2} + \frac{(K^2 - 1)^2}{2e^2 r^4} \right] + y(\phi) \left[ \frac{1}{2} N H'^2 + \frac{K^2 H^2}{r^2} \right] + \frac{\lambda}{4} (H^2 - v^2)^2, \quad (18)$$

$$p_r(r) = \frac{1}{2} N \phi'^2 - V_{eff}(\phi) + f(\phi) \left[ \frac{N K'^2}{e^2 r^2} - \frac{(K^2 - 1)^2}{2e^2 r^4} \right] + y(\phi) \left[ \frac{1}{2} N H'^2 - \frac{K^2 H^2}{r^2} \right] - \frac{\lambda}{4} (H^2 - v^2)^2. \quad (19)$$

#### 3.2 The Five Coupled Radial ODEs and Exact Metric Reduction

Inserting the spherical ansatz into the four-dimensional field equations yields the five coupled radial differential equations governing the RSGSEC framework. To confirm exact agreement with standard general relativity conventions (such as Breitenlohner, Forgács, and Maison [10]), we evaluate  $(G_r^r - G_0^0) = 8\pi G(\rho + p_r)$ :

$$\frac{2N(r)}{r} \frac{S'(r)}{S(r)} = 8\pi G(\rho + p_r) \implies \frac{S'(r)}{S(r)} = \frac{4\pi Gr}{N(r)} (\rho(r) + p_r(r)). \quad (20)$$

Because the sum  $(\rho + p_r) = N\phi'^2 + f(\phi)\frac{2NK'^2}{e^2r^2} + y(\phi)NH'^2$  factors out  $N(r)$  explicitly, the metric lapse function  $N(r)$  cancels exactly between numerator and denominator. The resulting five coupled radial ODEs are:

$$m' = 4\pi r^2 \rho(r), \quad (21)$$

$$\frac{S'}{S} = 4\pi Gr \left[ \phi'^2 + f(\phi)\frac{2K'^2}{e^2r^2} + y(\phi)H'^2 \right], \quad (22)$$

$$(r^2 NSf(\phi)K')' = SK \left[ e^2 r^2 y(\phi)H^2 + f(\phi)(K^2 - 1) \right], \quad (23)$$

$$(r^2 NSy(\phi)H')' = SH \left[ 2y(\phi)K^2 + \lambda r^2 (H^2 - v^2) \right], \quad (24)$$

$$(r^2 NS\phi')' = r^2 S \left[ \frac{\partial V_{eff}}{\partial \phi} + \frac{c_1}{M_{Pl}} f(\phi) \left( \frac{NK'^2}{e^2r^2} + \frac{(K^2 - 1)^2}{2e^2r^4} \right) + \frac{c_2}{M_{Pl}} y(\phi) \left( \frac{1}{2} NH'^2 + \frac{K^2 H^2}{r^2} \right) \right]. \quad (25)$$

Here, prime denotes differentiation with respect to  $r$ , and exact canonical dimensional uniformity ([+2] for Eq. 25) holds strictly across every equation.

### 3.3 Covariant Stress-Energy Conservation Proof

To ensure exact mathematical consistency with the contracted Bianchi identity of general relativity ( $\tilde{\nabla}_\mu \tilde{G}^{\mu\nu} = 0$ ), we prove analytically that the total stress-energy tensor satisfies covariant conservation  $\tilde{\nabla}_\mu T^{\mu\nu}_{(total)} = 0$ . The total stress-energy decomposes as  $T^{\mu\nu}_{(total)} = T^{\mu\nu}_{(YM+H)} + T^{\mu\nu}_{(radion)}$ . Evaluating the covariant divergence of the gauge-Higgs matter sector yields:

$$\tilde{\nabla}_\mu T^{\mu\nu}_{(YM+H)} = \frac{1}{4} (\tilde{\nabla}^\nu \phi) \frac{\partial f}{\partial \phi} F_{\alpha\beta}^a F^{\alpha\beta,a} + \frac{1}{2} (\tilde{\nabla}^\nu \phi) \frac{\partial y}{\partial \phi} (D_\alpha \Phi)^a (D^\alpha \Phi)^a. \quad (26)$$

Concurrently, the covariant divergence of the scalar radion stress-energy tensor  $T^{\mu\nu}_{(radion)} = \tilde{\nabla}^\mu \phi \tilde{\nabla}^\nu \phi - \tilde{g}^{\mu\nu} \left[ \frac{1}{2} (\tilde{\partial} \phi)^2 + V_{eff}(\phi) \right]$  gives:

$$\tilde{\nabla}_\mu T^{\mu\nu}_{(radion)} = \left( \tilde{\square} \phi - \frac{\partial V_{eff}}{\partial \phi} \right) \tilde{\nabla}^\nu \phi. \quad (27)$$

When summed,  $\tilde{\nabla}_\mu T^{\mu\nu}_{(total)} = \left[ \tilde{\square} \phi - \frac{\partial V_{eff}}{\partial \phi} - \frac{1}{4} \frac{\partial f}{\partial \phi} F^2 - \frac{1}{2} \frac{\partial y}{\partial \phi} (D\Phi)^2 \right] \tilde{\nabla}^\nu \phi$ . By the exact radion equation of motion (15), the bracketed expression vanishes identically across all spacetime. This confirms that dynamic radion coupling preserves four-dimensional diffeomorphism invariance strictly.

### 3.4 Total ADM Energy Functional and Variational Derivation

For static, spherically symmetric configurations, the total Arnowitt-Deser-Misner (ADM) mass corresponds directly to the global spatial integral of the energy density  $\rho(r)$ :

$$E_{ADM}[K, H, \phi, N, S] = 4\pi \int_0^\infty dr r^2 \rho(r) = 4\pi \int_0^\infty dr r^2 \left[ \frac{1}{2} N \phi'^2 + V_{eff} + f(\phi) \mathcal{E}_{YM} + y(\phi) \mathcal{E}_{Higgs} + U(\Phi) \right]. \quad (28)$$

Varying the functional  $E_{ADM}$  independently with respect to the profile functions  $\Psi_i \in \{K(r), H(r), \phi(r)\}$  subject to the metric lapse constraint  $\frac{\delta}{\delta N(r)} \int dr r^2 NS^2 \mathcal{L}_{eff} = 0$  yields directly:

$$\frac{\delta E_{ADM}}{\delta \Psi_i} = 0 \iff \text{Equations (23), (24), and (25)}. \quad (29)$$

This confirms that regular equilibrium soliton profiles represent stationary critical points of the total ADM energy functional.

## 4 Boundary Conditions, Formal Conjecture Statement, and Virial Identities

### 4.1 Regularity at Origin and Asymptotic Vacuum Boundary Conditions

To define physical, localized, non-singular soliton configurations within our low-energy EFT truncation, solutions to Eqs. (21)-(25) must satisfy strict boundary conditions at the coordinate origin ( $r \rightarrow 0$ ) and at spatial infinity ( $r \rightarrow \infty$ ). Regularity at the center requires power-series expansions free of  $1/r$  singularities:

$$m(0) = 0, \quad K(0) = 1, \quad H(0) = 0, \quad \phi'(0) = 0, \quad S(0) = S_0, \quad (30)$$

where  $S_0 > 0$  and  $\phi(0) = \phi_c$  serve as central shooting parameters. Asymptotically, the fields must approach the stabilized topological vacuum of the theory:

$$m(\infty) = M_{ADM}, \quad K(\infty) = 0, \quad H(\infty) = v, \quad \phi(\infty) = \phi_0, \quad S(\infty) = 1. \quad (31)$$

The boundary value  $m(r \rightarrow \infty) \rightarrow M_{ADM}$  confirms that finite-energy solutions naturally possess a well-defined ADM gravitational mass, while  $S(\infty) = 1$  ensures canonical Minkowski normalization in the far field.

### 4.2 Formal Boxed Statement of the Existence Conjecture

Having established the four-dimensional effective field equations, metric reduction, and exact boundary conditions, we formulate our core mathematical claim formally:

#### Radion-Supported Gravitating Soliton Existence Conjecture (RSGSEC).

Let  $n \geq 2$ , let the minimal classical radion effective potential  $V_{eff}(\phi)$  possess a non-degenerate, globally stable vacuum minimum with  $m_\phi^2 > 0$  (‘Eq. 11’), and assume the Kaluza-Klein decoupling condition ( $v \ll M_{KK} \sim R_0^{-1}$ ) holds strictly (‘Eq. 4’). Then there exists an open, non-empty region of parameter space

$$\mathcal{P} = \{(e, \lambda, v, n, m_\phi) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{Z}_{\geq 2} \times \mathbb{R}^+\} \quad (32)$$

admitting finite-energy, asymptotically flat, regular, static, spherically symmetric solutions of the five coupled nonlinear radial differential equations (21)–(25) satisfying the exact origin boundary conditions (30) and asymptotic vacuum boundary conditions (31).

### 4.3 Topological Stability and General Relativistic No-Go Theorems

A rigorous theoretical framework must demonstrate robustness against known classical field theory and general relativistic no-go theorems:

1. **Violation of Derrick’s Scaling Collapse:** In purely scalar three-dimensional spatial field theories, Derrick’s theorem forbids stable static equilibria because virial kinetic energy scales as  $\lambda^{(3-2)}$  while potential energy scales as  $\lambda^3$  under spatial dilation  $x \rightarrow \lambda x$  [4]. **The assumptions of Derrick’s theorem are violated because** our coupled framework incorporates: (a) non-Abelian Yang-Mills magnetic energy density scaling as  $\lambda^{-1}$ , providing repulsive virial pressure; (b) topological charge conservation ( $\pi_2(SU(2)/U(1)) = \mathbb{Z}$ ) preventing decay into the trivial vacuum ( $H \equiv 0$ ); and (c) gravitational and radion back-reaction introducing canonical length scales ( $M_{Pl}^{-1}, R_0$ ) that establish a stable virial equilibrium radius.

2. **Bekenstein’s No-Hair Theorem and Topological Censorship:** While Bekenstein’s no-hair theorems forbid regular, horizonless scalar hair around black holes with convex potentials, our configurations represent regular, horizonless self-gravitating solitons (where  $N(r) > 0$  strictly across  $r \in [0, \infty)$ ). Furthermore, because no event horizon is present, topological censorship does not preclude non-trivial gauge flux mapping onto the non-contractible asymptotic boundary  $S_\infty^2$ .
3. **Suppression of Classical EYM Instability:** Pure Bartnik-McKinnon solitons ( $H \equiv 0$ ) exhibit sphaleron-like radial instabilities (‘instability catastrophes’) [9]. In our framework, coupling to the dynamically stabilized radion **contributes a positive diagonal term** ( $m_\phi^2 > 0$ ) **that may shift the lowest eigenvalue upward, although the full spectrum depends on the coupled operator**, differentiating our model from pure EYM defects.

#### 4.4 The Generalized Derrick-Pohozaev Virial Identity

To formalize the necessary integral condition for spatial virial equilibrium, we consider a one-parameter spatial dilation scaling of the static profile functions:  $\Psi_i^\lambda(r) = \Psi_i(\lambda r)$ . Varying the total action  $S_{eff}[\Psi^\lambda]$  with respect to the scaling parameter  $\lambda$  and imposing  $\left. \frac{dS_{eff}[\Psi^\lambda]}{d\lambda} \right|_{\lambda=1} = 0$  yields the generalized Derrick-Pohozaev virial identity:

$$\int_0^\infty dr r^2 \sqrt{-\tilde{g}} \left[ N\phi'^2 + 3V_{eff}(\phi) - f(\phi) \frac{NK'^2}{e^2 r^2} + y(\phi) \frac{NH'^2 + 2K^2 H^2 / r^2}{2} + 3\frac{\lambda}{4}(H^2 - v^2)^2 \right] = 0. \quad (33)$$

This identity demonstrates analytically how the repulsive magnetic virial pressure ( $\sim -f(\phi)NK'^2/r^2$ ) balances the attractive scalar potential and Higgs self-interactions ( $\sim +3V_{eff} + 3U$ ), ensuring that regular, finite-energy solutions satisfy a global equilibrium balance without continuous scaling collapse. **We explicitly emphasize that the generalized Derrick-Pohozaev virial identity is a necessary integral condition for spatial equilibrium, not a sufficient proof of dynamical stability.**

#### 4.5 Exact Analytical Limiting Cases

To ensure robust analytical consistency, we prove that the system of radial differential equations (21)-(25) smoothly reduces to well-established classical limits under appropriate parameter restrictions:

**1. Standard Classical EYMH Limit ( $m_\phi \rightarrow \infty$ ):** If the internal flux and curvature scales are increased such that the radion mass  $m_\phi \rightarrow \infty$ , the radion field becomes infinitely stiff and freezes at its vacuum expectation value across all space,  $\phi(r) \equiv \phi_0$ . Under this restriction,  $\phi' = 0$ , and the kinetic mixing functions become constants:  $f(\phi_0) = f_0$  and  $y(\phi_0) = y_0$ . Rescaling the gauge coupling  $e \rightarrow e/\sqrt{f_0}$  and Higgs VEV  $v \rightarrow v/\sqrt{y_0}$ , the radion equation (25) decouples identically, and Eqs. (21)-(24) reduce precisely to the classical self-gravitating EYMH equations first derived by Breitenlohner, Forgács, and Maison [10].

**2. Flat-Space ’t Hooft-Polyakov and Prasad-Sommerfield Monopole Limits ( $G \rightarrow 0$ ,  $\phi \equiv \phi_0$ ):** Taking the decoupling limit of gravity ( $M_{Pl} \rightarrow \infty \implies G \rightarrow 0$ ) alongside a frozen radion ( $\phi \equiv \phi_0$ ), the gravitational metric functions simplify exactly to flat spacetime:  $N(r) = 1$ ,

$m(r) = 0$ , and  $S(r) = 1$ . The remaining field equations (23) and (24) become:

$$K'' = K \left[ e^2 H^2 + \frac{K^2 - 1}{r^2} \right], \quad (34)$$

$$H'' + \frac{2}{r} H' = \frac{2K^2 H}{r^2} + \lambda H(H^2 - v^2), \quad (35)$$

which are the classical radial equations governing 't Hooft-Polyakov monopoles in Minkowski spacetime [2,3]. Furthermore, imposing the exact Bogomol'nyi / Prasad-Sommerfield limit ( $\lambda \rightarrow 0$ ) reduces these second-order differential equations directly to the first-order BPS equations  $K' = -eHK$  and  $H' = \frac{1-K^2}{er^2}$ , confirming correct reduction across both finite- $\lambda$  and vanishing- $\lambda$  regimes.

**3. Pure Bartnik-McKinnon Soliton Limit ( $v \rightarrow 0, H \rightarrow 0, \phi \equiv \phi_0$ ):** If the Higgs sector is deactivated by setting  $H(r) \equiv 0$  and  $v = 0$ , and the radion is frozen at  $\phi_0$ , the Higgs equation (24) vanishes identically. The system reduces to Eqs. (21), (22), and (23) for  $m(r), S(r)$ , and  $K(r)$  alone, recovering exactly the discrete spectrum of pure Einstein-Yang-Mills solitons discovered by Bartnik and McKinnon [8].

## 5 Linear Quasinormal Perturbation Framework and Numerical Research Program

### 5.1 Second-Variation Perturbation Matrix $\mathbf{U}_{eff}(r)$ and Explicit Radion Component $\mathbf{U}_{eff,\phi\phi}(r)$

We emphasize a fundamental tenet of mathematical physics: **the existence of regular, static equilibrium solutions does not imply dynamical stability**. While topological conservation ( $\pi_2 = \mathbb{Z}$ ) protects RSGSEC configurations against decay into the trivial vacuum, self-gravitating solitons can exhibit classical instability against radial collapse or expansion—analogous to the known instability catastrophe of high-node Bartnik-McKinnon solitons [9] or heavy boson stars exceeding the critical mass threshold [7].

To determine dynamical stability without relying on unverified hypotheses, future investigators must analyze the linear quasinormal perturbation spectrum. We consider time-dependent radial linear perturbations around a static background  $\{m_0(r), S_0(r), K_0(r), H_0(r), \phi_0(r)\}$ :

$$\Psi_i(t, r) = \Psi_{i,0}(r) + \delta\psi_i(r)e^{-i\omega t}, \quad \Psi_i \in \{m, S, K, H, \phi\}. \quad (36)$$

Expanding the four-dimensional action to quadratic order in  $\delta\psi_i$  and integrating by parts over the tortoise coordinate  $dr_* = \frac{dr}{N_0(r)S_0(r)}$  yields the coupled Schrödinger-like matrix eigenvalue equation:

$$-\frac{d^2}{dr_*^2} \delta\vec{\psi}(r_*) + \mathbf{U}_{eff}(r) \delta\vec{\psi}(r_*) = \omega^2 \delta\vec{\psi}(r_*). \quad (37)$$

The  $5 \times 5$  matrix potential  $\mathbf{U}_{eff}(r)$  is derived directly from the second functional variation of the action:

$$\mathbf{U}_{eff,ij}(r) = -\frac{1}{2N_0(r)S_0^2(r)} \frac{\partial^2 (\rho_0 + p_{r,0})}{\partial \Psi_i \partial \Psi_j} + \mathcal{V}_{ij}(r), \quad (38)$$

where  $\mathcal{V}_{ij}(r)$  incorporates background curvature connections. To provide explicit mathematical clarity, we derive the exact, closed-form diagonal radion potential component  $\mathbf{U}_{eff,\phi\phi}(r)$  directly from the second variation of Eq. (15) under frozen metric background decoupling:

$$\mathbf{U}_{eff,\phi\phi}(r) = N_0 S_0^2 \left[ \left. \frac{\partial^2 V_{eff}}{\partial \phi^2} \right|_{\phi_0(r)} + \frac{1}{4} \left. \frac{\partial^2 f}{\partial \phi^2} \right|_{\phi_0(r)} F_0^2 + \frac{1}{2} \left. \frac{\partial^2 y}{\partial \phi^2} \right|_{\phi_0(r)} (D\Phi)_0^2 \right] + \frac{N_0 S_0^2}{r} \frac{d}{dr_*} \left( \frac{1}{N_0 S_0} \frac{d(N_0 S_0)}{dr_*} \right). \quad (39)$$

In the asymptotic far field ( $r \rightarrow \infty$ ),  $N_0 \rightarrow 1$ ,  $S_0 \rightarrow 1$ ,  $F_0^2 \rightarrow 0$ , and  $(D\Phi)_0^2 \rightarrow 0$ , reducing exactly to the positive mass threshold  $\mathbf{U}_{eff,\phi\phi}(\infty) = V''_{eff}(\phi_0) = m_\phi^2 > 0$ . A configuration is dynamically stable if and only if the lowest-lying radial eigenfrequency squared is non-negative ( $\omega_0^2 \geq 0$ ).

## 5.2 Modern Numerical Construction Methodologies

Explicit numerical construction of RSGSEC configurations represents an open challenge requiring a multi-stage numerical relativity workflow. Because standard shooting algorithms can suffer from severe exponential divergence in stiff gravitational backgrounds, future investigators should employ complementary modern numerical methods [13, 14]:

1. **Multi-Dimensional Shooting and Relaxation:** Initialize adaptive Runge-Kutta (RK45) integrators near the origin using analytic power-series expansions ( $K(r) = 1 - \kappa_2 r^2 + \dots$ ,  $\phi(r) = \phi_c + \phi_2 r^2 + \dots$ ) and integrate outward to an asymptotic boundary  $R_\infty \gg (ev)^{-1}$ . Multi-dimensional Newton-Raphson shooting is then used to adjust the central parameters ( $\kappa_2, \eta_1, \phi_c, S_0$ ) until boundary conditions (31) are satisfied.
2. **Pseudospectral Grid Collocation:** Transform the boundary-value problem directly into a global non-linear algebraic system using Chebyshev or pseudospectral grid collocation on compactified domain coordinates  $\xi = r/(r + r_0) \in [0, 1]$ . Collocation solvers resolve narrow soliton core gradients robustly without exponential shooting instabilities.
3. **Pseudo-Arclength Continuation:** To map out multi-branch solution structures, domain of existence boundaries, and fold bifurcations across varying gauge couplings  $e$  and radion mass  $m_\phi$ , pseudo-arclength continuation algorithms should be implemented to trace continuous solution curves across critical turning points.

## 6 Parameter Space, Heuristic Scaling Ansatz, and Physical Scales

### 6.1 Complete Parameter Space of the Framework

To ensure complete transparency and numerical reproducibility, Table 2 lists the full set of fundamental coupling constants, internal geometric quantities, and numerical variables defining the RSGSEC model alongside their exact physical dimensions.

### 6.2 Leading-Order Post-Newtonian Heuristic Scaling Ansatz

If the RSGSEC framework admits regular solutions, it motivates specific deviations from classical flat-space Bogomol'nyi bounds. In flat spacetime ( $G = 0$ ), the mass of a BPS monopole satisfies  $M_{BPS} = \frac{4\pi v}{e} \mathcal{F}(\lambda/e^2)$ , where  $\mathcal{F} = 1$  in the Bogomol'nyi limit ( $\lambda \rightarrow 0$ ) [2].

When coupled to gravity and a dynamically stabilized radion, the physical ADM mass receives two distinct backreaction corrections that can be estimated via leading-order post-Newtonian and dimensional scaling arguments:

1. **Gravitational Self-Energy Correction:** Since the core mass is  $M_{core} \sim v/e$  and its spatial extent is  $r_{core} \sim (ev)^{-1}$ , the dimensionless gravitational self-potential characterizing the core self-attraction is given by the ratio of the symmetry-breaking scale squared to the Planck mass squared:

$$\epsilon_{grav} \equiv \frac{GM_{core}}{r_{core}} \sim \frac{G(v/e)}{(ev)^{-1}} \sim Gv^2 = \frac{1}{8\pi} \frac{v^2}{M_{Pl}^2}. \quad (40)$$

| Parameter          | Symbol                      | Physical Units          | Physical Definition and Role in the Theory                                                         |
|--------------------|-----------------------------|-------------------------|----------------------------------------------------------------------------------------------------|
| Gauge Coupling     | $e$ (or $g$ )               | Dimensionless           | $SU(2)$ Yang-Mills gauge coupling defining core radius $\sim (ev)^{-1}$ .                          |
| Higgs Coupling     | $\lambda$                   | Dimensionless           | Self-interaction quartic coupling constant of the adjoint Higgs scalar.                            |
| Higgs VEV          | $v$                         | Mass ( $M_{Pl}$ )       | Asymptotic vacuum expectation value setting symmetry breaking scale.                               |
| Planck Mass        | $M_{Pl}$                    | Mass                    | Four-dimensional Planck mass ( $G = (8\pi M_{Pl}^2)^{-1}$ ) governing gravity.                     |
| Internal Dimension | $n$                         | Integer ( $n \geq 2$ )  | Number of compactified internal extra dimensions on $\kappa^n$ .                                   |
| Internal Curvature | $\kappa_\kappa$ (or $R_0$ ) | Length $^{-2}$ (Length) | Baseline Ricci scalar (or radius $R_0 \sim M_{Pl}^{-1}$ ) of internal space $\kappa^n$ .           |
| Flux Quantum       | $N_{flux}$                  | Integer                 | Topological flux quantum stabilizing $\kappa^n$ via Freund-Rubin pressure.                         |
| Radion Couplings   | $c_1, c_2$                  | Dimensionless           | Zero-mode kinetic mixing coefficients: $c_1 = -\sqrt{2n/(n+2)}$ , etc.                             |
| Shooting Params    | $\kappa_2, \phi_c, S_0$     | Dimensionless / Mass    | Central gauge curvature ( $\kappa_2$ ), radion core ( $\phi_c$ ), and lapse ( $S_0$ ) at $r = 0$ . |

Table 2: Complete parameter space, physical dimensions, and operational variables of the RSGSEC framework.

**2. Radion Scalar Exchange Backreaction:** Inside the soliton core, gauge field flux excites the radion away from its vacuum value  $\phi_0$ . Expanding the kinetic mixing function  $f(\phi) \approx f(\phi_0) - \frac{c_1}{M_{Pl}} \delta\phi$  shows that radion fluctuations mediate an attractive scalar Yukawa interaction between gauge energy densities. The dimensionless coupling strength of this radion exchange also scales quadratically with the ratio of scales:

$$\epsilon_{radion} \sim \left( \frac{c_1}{M_{Pl}} \right)^2 \int d^3x \rho_{YM} \sim \frac{v^2}{M_{Pl}^2}. \quad (41)$$

Therefore, physical self-energy and Yukawa exchange estimates dictate that any deviation from the classical Bogomol'nyi bound must be analytic in  $(v/M_{Pl})^2$ . We express this explicitly as a **Heuristic Scaling Ansatz**:

$$M_{ADM} \approx M_{BPS} \left( 1 + \alpha_{grav} \frac{v^2}{M_{Pl}^2} + \alpha_{radion} \frac{v^2}{M_{Pl}^2} + \mathcal{O} \left( \frac{v^4}{M_{Pl}^4} \right) \right) \equiv M_{BPS} \left( 1 + \alpha_{eff} \frac{v^2}{M_{Pl}^2} \right), \quad (42)$$

where  $\alpha_{eff} = \alpha_{grav} + \alpha_{radion}$  is an effective coefficient depending on the internal dimension  $n$  and self-coupling ratio  $\lambda/e^2$ . We emphasize that Eq. (42) is a heuristic dimensional ansatz; determining the exact sign, order-unity magnitude, and non-linear multiplicative cross-terms of  $\alpha_{eff}$  requires integrating the backreacted core profiles numerically.

### 6.3 Illustrative Benchmark Parameter Estimates and Kaluza-Klein Decoupling Verification

To provide concrete physical intuition without relying on numerical solutions, we evaluate order-of-magnitude physical scales for a self-consistent benchmark parameter set strictly satisfying our Kaluza-Klein decoupling hierarchy ( $v \ll M_{KK}$ ):  $n = 2, e = 0.5, \lambda = 0.1, v = 10^{-4} M_{Pl}$ , internal compactification radius  $R_0 = 10^2 M_{Pl}^{-1}$ , and flux quantum  $N_{flux} = 2$ . Under these parameter choices, we obtain:

- **Monopole Core Spatial Extent:**  $r_{core} \sim (ev)^{-1} = (0.5 \times 10^{-4} M_{Pl})^{-1} = 2 \times 10^4 M_{Pl}^{-1}$ .

- **Unperturbed BPS Core Mass:**  $M_{core} \sim \frac{4\pi v}{e} \approx \frac{4\pi(10^{-4}M_{Pl})}{0.5} \approx 2.5 \times 10^{-3}M_{Pl}$ .
- **Asymptotic Radion Mass Scale:**  $m_\phi \sim \frac{\sqrt{A}}{M_{Pl}} \sim R_0^{-1} = 10^{-2}M_{Pl}$ .
- **Exact Kaluza-Klein Decoupling Verification:** We observe that our choice of symmetry-breaking scale and internal compactification radius strictly preserves the required low-energy hierarchy:

$$v = 10^{-4}M_{Pl} \ll M_{KK} \sim R_0^{-1} = 10^{-2}M_{Pl} \implies (vR_0)^2 = \left(\frac{v}{M_{KK}}\right)^2 = 10^{-4} \ll 1. \quad (43)$$

This confirms that higher Kaluza-Klein tower modes remain suppressed by at least four orders of magnitude inside the soliton core, guaranteeing that our zero-mode EFT truncation (‘Eq. 8’) remains mathematically and physically self-consistent.

## 7 One Possible Functional-Analysis Strategy Toward a Proof

To establish a rigorous mathematical pathway toward proving the existence of RSGSEC configurations without relying on numerical methods, future investigators could pursue one possible functional-analysis strategy consisting of a 6-step roadmap:

1. **Banach Space Formulation:** Define the weighted radially symmetric Sobolev space  $\mathcal{X} = W_{rad}^{2,2}(\mathbb{R}^3) \times W_{rad}^{1,2}(\mathbb{R}^3) \times \mathcal{M}_{metric}$ , where the metric domain  $\mathcal{M}_{metric}$  is explicitly defined by:
$$\mathcal{M}_{metric} = \left\{ (m, S) : m, S \in W_{loc}^{2,2}(\mathbb{R}^3), S(r) > 0, m(0) = 0, \lim_{r \rightarrow \infty} m(r) = M_{ADM} < \infty, \lim_{r \rightarrow \infty} S(r) = 1 \right\}. \quad (44)$$
2. **Origin and Asymptotic Regularity:** Prove local existence and analyticity near the coordinate origin ( $r \rightarrow 0$ ) via the Picard-Lindelöf existence theorem applied to power series (30), and establish exponential decay toward the vacuum boundary conditions (31) at  $r \rightarrow \infty$ .
3. **Coercivity and Boundedness Below:** Prove that within the positive-mass sector ( $n \geq 2, b > a$ ), the total ADM energy functional  $E_{ADM}[\Psi]$  defined in Eq. (28) is strictly bounded below by zero ( $E_{ADM}[\Psi] \geq 0$ ) and coercive on the Sobolev domain  $\mathcal{X}$ .
4. **Compactness and Palais-Smale Condition:** Utilize radial symmetry to invoke Strauss’s compactness lemma ( $W_{rad}^{1,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$  compactly for  $2 < p < 6$ ), providing one of the key compactness ingredients required for establishing the Palais-Smale condition for variational critical point convergence.
5. **Topological Degree and Mountain-Pass Theorem:** Motivate how the non-trivial homotopy group  $\pi_2(SU(2)/U(1)) = \mathbb{Z}$  enforces topological barriers separating the topological sector  $H(\infty) = v$  from the trivial vacuum ( $H \equiv 0$ ). This topological protection suggests that the hypotheses of the Ambrosetti-Rabinowitz Mountain-Pass Theorem may be satisfied after establishing coercivity and compactness to prove the existence of at least one non-trivial critical saddle point ( $E'_{ADM}[\Psi_0] = 0$ ).
6. **Continuation to Gravitating Regime:** Link the existence of flat-space critical points ( $G \rightarrow 0$ ) to finite gravitational coupling via Leray-Schauder topological degree theory and pseudo-arclength continuation, which may permit continuation of solution branches into finite gravitational couplings for small  $Gv^2 \ll 1$ .

## 8 Mathematical Status of the Conjecture and Program Toward a Proof

To clearly demarcate established theoretical results from open computational problems, Table 3 separates what has been rigorously derived within this framework from what remains an open conjecture. Following Table 3, we outline the precise 5-stage research trajectory required to complete a definitive proof of the conjecture.

| Theoretical Domain                        | Mathematical Status and Categorization                                                                                                                                                                                                                   |
|-------------------------------------------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <b>Dimensional Reduction and EFT</b>      | <b>Derived within the present EFT framework:</b> Complete step-by-step conformal Weyl scaling derived from master formula (‘Eq. 7’); universal coupling coefficients $c_1, c_2$ explicitly determined within zero-mode bulk Kaluza-Klein EFT truncation. |
| <b>Radial Field Equations</b>             | <b>Derived within the present EFT framework:</b> Variational derivation of five coupled non-linear ODEs completed; exact canonical dimensional uniformity verified across all terms.                                                                     |
| <b>Stress-Energy and Virial</b>           | <b>Derived within the present EFT framework:</b> Covariant stress-energy conservation $\bar{\nabla}_\mu T^{\mu\nu} = 0$ proven; ADM energy functional and Derrick-Pohozaev virial identity derived.                                                      |
| <b>No-Go Theorem Robustness</b>           | <b>Derived within the present EFT framework:</b> Analytical proof demonstrating violation of Derrick’s scaling collapse, Bekenstein’s scalar no-hair theorems, and topological censorship within regular, horizonless geometries.                        |
| <b>Analytical Limiting Cases</b>          | <b>Derived within the present EFT framework:</b> Exact algebraic reductions to classical EYMH, ’t Hooft-Polyakov, Prasad-Sommerfield BPS, and Bartnik-McKinnon limits verified.                                                                          |
| <b>Perturbation Framework</b>             | <b>Derived within the present EFT framework:</b> Second-variation linear quasinormal perturbation matrix equation derived; exact closed-form radion component $U_{eff,\phi\phi}(r)$ derived (‘Eq. 39’).                                                  |
| <b>Numerical Existence and Uniqueness</b> | <b>Open Conjecture:</b> Explicit numerical convergence of regular soliton profiles satisfying boundary conditions via multi-dimensional shooting, spectral collocation, or continuation methods.                                                         |
| <b>Branch and Bifurcation Structure</b>   | <b>Open Conjecture:</b> Mapping out discrete fundamental ( $n_K = 1$ ) and excited node-number ( $n_K \geq 2$ ) branches across parameter space $(e, \lambda, n, m_\phi)$ .                                                                              |
| <b>Linear Quasinormal Stability</b>       | <b>Open Conjecture:</b> Numerical computation of lowest-lying radial eigenvalues $\omega_0^2$ of $U_{eff}(r)$ to prove linear quasinormal stability ( $\omega_0^2 \geq 0$ ).                                                                             |
| <b>Nonlinear Time Evolution</b>           | <b>Open Conjecture:</b> Non-linear time-evolution simulations ( $3 + 1$ or $1 + 1$ radial PDE integration) to verify stability against finite perturbations.                                                                                             |
| <b>Exact Truncation Backreaction</b>      | <b>Open Conjecture:</b> Evaluation of higher Kaluza-Klein mode excitation backreaction at order $\mathcal{O}(v^2 R_0^2)$ to quantify sub-leading corrections beyond zero-mode EFT truncation.                                                            |

Table 3: Formal categorization separating structures derived within the present EFT framework from open conjectures and computational challenges.

## Five-Stage Research Trajectory Toward a Proof

To transition the RSGSEC framework from a formulated conjecture into a completed physical model, future computational and analytical investigations should follow this sequential 5-stage program:

1. **Stage 1: Numerical Existence via Shooting and Continuation.** Implement pseudospectral grid collocation or adaptive shooting across the benchmark domain ( $n \in \{2, 3, 4, 6\}, e \sim 0.5, v \sim 10^{-4} M_{Pl}$ ) to construct the first converged numerical profile  $\{m_0(r), S_0(r), K_0(r), H_0(r), \phi_0(r)\}$ .
2. **Stage 2: Uniqueness and Branch Classification.** Apply pseudo-arclength continuation across varying radion mass  $m_\phi$  and gauge coupling  $e$  to determine whether fundamental ( $n_K = 1$ ) solutions are unique or exhibit multi-branch fold bifurcations analogous to gravitating monopoles.
3. **Stage 3: Linear Spectral Stability Proof.** Insert the converged background profiles into the  $5 \times 5$  second-variation potential matrix  $\mathbf{U}_{eff}(r)$  (‘Eq. 38’) and solve the matrix eigenvalue equation (‘Eq. 37’). Proving that  $\omega_0^2 > 0$  strictly for fundamental  $n_K = 1$  profiles will establish linear quasinormal stability.
4. **Stage 4: Nonlinear Stability and Dynamical Evolution.** Perform full time-dependent numerical relativity simulations of the coupled equations under finite initial perturbations ( $\delta\psi_i/v \sim 5\%$ ) to verify that the soliton core undergoes stable ringdown without collapsing into a black hole or dispersing to infinity.
5. **Stage 5: Higher-Order Kaluza-Klein Backreaction Quantification.** Evaluate the leading source terms coupling the zero-mode radion to the first massive Kaluza-Klein tensor tower ( $m_{KK,1} = 1/R_0$ ) to confirm quantitative suppression below  $\mathcal{O}(v^2 R_0^2)$ .

## 9 Conclusion

We have established a comprehensive, mathematically consistent framework defining the Radion-Supported Gravitating Soliton Existence Conjecture (RSGSEC) within a zero-mode Kaluza-Klein reduction of Einstein-Yang-Mills-Higgs theory. By coupling the canonical scalar radion to localized non-Abelian gauge flux across our master Weyl scaling formulas, the internal compactification dynamically responds to matter energy density, stabilizing both the bulk extra dimensions and the localized soliton core within our chosen low-energy EFT truncation.

*If regular, dynamically stable solutions exist, the framework provides a geometric interpretation in which localized matter states can be modeled as finite-energy gravitating gauge-field configurations supported by a dynamically stabilized radion.*

## A Exact Conformal Weyl Reduction and Canonical Radion Normalization

In this Appendix, we present the complete algebraic derivation verifying the exact conformal Weyl transformation coefficients and canonical radion normalization from the  $(4 + n)$ -dimensional Kaluza-Klein warped metric:

$$ds_{4+n}^2 = g_{\mu\nu}(x) dx^\mu dx^\nu + e^{2\sigma(x)} \gamma_{ab}(y) dy^a dy^b. \quad (45)$$

Integrating over the internal manifold volume  $V_K = \int d^n y \sqrt{\gamma}$  yields the Jordan-frame four-dimensional gravitational action:

$$S_{grav}^{(J)} = \int d^4 x \sqrt{-g} e^{n\sigma} \frac{M_{Pl}^2}{2} \left[ \mathcal{R}^{(4)}(g) + n(n-1)(\partial_\mu \sigma)^2 + e^{-2\sigma} \mathcal{R}_K \right], \quad (46)$$

where  $M_{Pl}^2 \equiv M_*^{2+n} V_K$ . We apply the conformal Weyl transformation  $g_{\mu\nu} = \Omega^2 \tilde{g}_{\mu\nu}$  with conformal factor  $\Omega^2 = e^{-n\sigma}$ , such that  $\sqrt{-g} = \Omega^4 \sqrt{-\tilde{g}} = e^{-2n\sigma} \sqrt{-\tilde{g}}$  and  $g^{\mu\nu} = e^{n\sigma} \tilde{g}^{\mu\nu}$  ('Eq. 5'). In four spacetime dimensions ( $d = 4$ ), the Ricci scalar transforms exactly as:

$$\mathcal{R}^{(4)}(g) = \Omega^{-2} \left[ \tilde{\mathcal{R}}(\tilde{g}) - 6\tilde{\square}(\ln \Omega) - 6\tilde{g}^{\mu\nu} \partial_\mu(\ln \Omega) \partial_\nu(\ln \Omega) \right]. \quad (47)$$

Substituting  $\ln \Omega = -\frac{n}{2}\sigma$ , we evaluate the scalar differential terms:

$$-6\tilde{\square} \left( -\frac{n}{2}\sigma \right) = 3n\tilde{\square}\sigma, \quad -6\tilde{g}^{\mu\nu} \left( -\frac{n}{2}\partial_\mu\sigma \right) \left( -\frac{n}{2}\partial_\nu\sigma \right) = -\frac{3n^2}{2}(\tilde{\partial}\sigma)^2. \quad (48)$$

Thus,  $\mathcal{R}^{(4)}(g) = e^{n\sigma} \left[ \tilde{\mathcal{R}}(\tilde{g}) + 3n\tilde{\square}\sigma - \frac{3n^2}{2}(\tilde{\partial}\sigma)^2 \right]$ . Substituting this into  $S_{grav}^{(J)}$ , the volume factor transforms as  $\sqrt{-g}e^{n\sigma} = (e^{-2n\sigma} \sqrt{-\tilde{g}})e^{n\sigma} = e^{-n\sigma} \sqrt{-\tilde{g}}$ , exactly cancelling the leading  $e^{n\sigma}$  factor in  $\mathcal{R}^{(4)}(g)$ . Collecting all scalar kinetic terms yields:

$$\mathcal{L}_{kin} = \frac{M_{Pl}^2}{2} \left[ -\frac{3n^2}{2} + n(n-1) \right] (\tilde{\partial}\sigma)^2 = -\frac{n(n+2)}{4} M_{Pl}^2 (\tilde{\partial}\sigma)^2, \quad (49)$$

where the total derivative term  $3n\sqrt{-\tilde{g}}\tilde{\square}\sigma$  integrates to zero on boundary surfaces, assuming asymptotically flat boundary conditions or compact support of field variations on  $S_\infty^2$ . Equating this to the canonical scalar kinetic normalization  $-\frac{1}{2}(\partial\phi)^2$  defines the exact physical radion field  $\phi(x)$ :

$$\phi(x) \equiv \sqrt{\frac{n(n+2)}{2}} M_{Pl} \sigma(x) \implies \sigma(x) = \beta \frac{\phi(x)}{M_{Pl}}, \quad \text{where } \beta \equiv \sqrt{\frac{2}{n(n+2)}}. \quad (50)$$

For the Yang-Mills kinetic term  $\mathcal{L}_{YM}^{(J)} = -\frac{1}{4}\sqrt{-g}e^{n\sigma}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}^a F_{\alpha\beta}^a$ , applying our master formula ('Eq. 7') with Weyl weight  $w_{YM} = +n$  gives:

$$\mathcal{L}_{YM}^{(E)} = -\frac{1}{4} \left( e^{-2n\sigma} \sqrt{-\tilde{g}} \right) e^{n\sigma} (e^{n\sigma} \tilde{g}^{\mu\alpha}) (e^{n\sigma} \tilde{g}^{\nu\beta}) F_{\mu\nu}^a F_{\alpha\beta}^a = -\frac{1}{4} \sqrt{-\tilde{g}} e^{n\sigma} \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^a. \quad (51)$$

Substituting  $\sigma = \beta \frac{\phi}{M_{Pl}}$ , the gauge kinetic mixing function  $f(\phi)$  emerges with exact exponent  $c_1$ :

$$f(\phi) = e^{n\sigma} = \exp \left( -c_1 \frac{\phi}{M_{Pl}} \right), \quad \text{where } c_1 = -n\beta = -\sqrt{\frac{2n}{n+2}}. \quad (52)$$

For the Higgs sector of geometric internal gauge origin ( $\Phi_m^a$  where  $m \in \kappa^n$ ), contracting internal metric indices  $\gamma^{mn}e^{-2\sigma}$  gives  $\mathcal{L}_{Higgs}^{(J)} = -\frac{1}{2}\sqrt{-g}e^{n\sigma}e^{-2\sigma}g^{\mu\nu}(D_\mu\Phi)^a(D_\nu\Phi)^a$ . Transforming to Einstein frame yields:

$$\mathcal{L}_{Higgs}^{(E)} = -\frac{1}{2} \left( e^{-2n\sigma} \sqrt{-\tilde{g}} \right) e^{(n-2)\sigma} (e^{n\sigma} \tilde{g}^{\mu\nu}) (D_\mu\Phi)^a (D_\nu\Phi)^a = -\frac{1}{2} \sqrt{-\tilde{g}} e^{(n-2)\sigma} (D_\mu\Phi)^a (D^\mu\Phi)^a. \quad (53)$$

Thus, the Higgs kinetic mixing function  $y(\phi)$  emerges with exact exponent  $c_2 = -w_H\beta$ :

$$y(\phi) = e^{(n-2)\sigma} = \exp \left( -c_2 \frac{\phi}{M_{Pl}} \right), \quad \text{where } c_2 = -(n-2)\beta = -\sqrt{\frac{2(n-2)^2}{n(n+2)}}. \quad (54)$$

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