Topological Photonic Sheaves and Bulk Gauge Continuity: A $(4+k)$-Dimensional Geometric Extension of Maxwell Electrodynamics

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Abstract

This work contains some solutions for the category errors, degree mismatches, and differential geometry of the original $U(1)$ toy model that was described in our earlier Maxwell work.

We formulate a geometric extension of abelian gauge theory by embedding four-dimensional spacetime X into a smooth, oriented fiber bundle $\pi : Y \to X$ with a compact, boundaryless k-dimensional Riemannian manifold K as fiber ($\dim_{\mathbb{R}} Y = 4 + k$) [1]. To maintain categorical consistency, we model the bulk space Y as a complex analytic manifold with even fiber dimension $k \in 2\mathbb{N}$, letting $\mathcal{E}_P = \mathcal{O}_Y(\mathcal{L}) \in \mathrm{Coh}(Y)$ denote the coherent sheaf of holomorphic sections of a holomorphic line bundle $\mathcal{L} \to Y$ [1]. The underlying smooth line bundle supports a $U(1)$ gauge connection locally represented by potentials $\mathcal{A}$ with globally defined curvature $\mathcal{F} \in \Omega^2_{\mathrm{cl}}(Y)$ [1]. Coupling $\mathcal{E}_P$ to a bulk Higgs sheaf $\mathcal{H} \in \mathrm{Coh}(Y)$, we formulate the interaction via the derived extension class $\eta \in \mathrm{Ext}^{k+3}_Y(\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H}, \mathcal{O}_Y)$ [1].

Through a postulated de Rham realization map $\mathfrak{c}_\eta$, the class η determines a closed differential form $\mathcal{J} \in \Omega^{k+3}_{\mathrm{cl}}(Y)$ satisfying $d_Y \mathcal{J} = 0$ [1]. Fiber integration along K induces a closed 3-form current on physical spacetime:

$$J_s := \pi_* \mathcal{J} = \int_K \mathcal{J} \in \Omega^3(X), \qquad d_X J_s = \pi_*(d_Y \mathcal{J}) = 0. \tag{1}$$

The degree-consistent extended Maxwell equations are formulated as $dF = 0$ and $d{*}F = J_s$, where $F \in \Omega^2(X)$ is the physical electromagnetic field strength [1,4]. For any spatial 3-chain $\Sigma^3 \subset X$ with boundary $\partial\Sigma^3$, this induces a localized Gauss-law topological charge:

$$Q_{\mathrm{soul}}(\Sigma^3) = \int_{\Sigma^3} J_s = \int_{\partial\Sigma^3} {*}F. \tag{2}$$

We impose the explicit decoupling condition $\mathcal{J}_{k=0} \equiv 0$, guaranteeing that standard classical electrodynamics is recovered identically on uncompactified four-dimensional manifolds ($k = 0$) [1].

Coupling J_s to the gauge potential via the effective action $S_{\mathrm{eff}}[A] = -\frac{1}{4}\int_X F \wedge {*}F + \int_X A \wedge J_s$ preserves 4D gauge invariance under $A \mapsto A + d\lambda$ as a direct consequence of current closure $d_X J_s = 0$ [1]. Under compactification on an isotropic torus of characteristic scale Λ, we formulate a phenomenological scaling hypothesis for multi-path quantum interference visibility:

$$1 - V \approx \frac{1}{2}\langle (\Delta\phi_s)^2 \rangle \sim \|\omega(\eta)\|^2 \Lambda^{-2k}. \tag{3}$$

We structure the differential-cohomological lifting requirements with integral periods [2], formulate the anomaly-consistency conditions required of any complete bulk spectrum, and outline experimental detection bounds from precision optical interferometry.

1 Introduction

Classical electrodynamics in four-dimensional Minkowski spacetime X is governed by the Maxwell equations $dF = 0$ and $d * F = 0$, where $F = dA \in \Omega^2(X)$ is the electromagnetic curvature 2-form [4]. In the Standard Model, these equations enforce exact $U(1)_{\mathrm{em}}$ gauge invariance and ensure photon masslessness down to the electroweak scale. However, attempts to unify electroweak interactions within higher-dimensional supergravity, string compactifications, and derived algebraic geometry suggest that low-energy gauge sectors may represent projections of richer topological structures supported on higher-dimensional bulk spaces Y [1, 3].

In this paper, we formulate a geometric extension of Maxwell electrodynamics termed *Photon Soul Continuity* [1]. Rather than introducing ad-hoc phenomenological vector fields, we propose a categorical pipeline linking derived coherent sheaf morphisms on a $(4 + k)$-dimensional bulk space Y to closed differential forms on the physical 4D spacetime X [1, 2].

The paper is structured as follows: Section 2 defines the underlying geometric fiber bundle, sheaf categories, and the derived extension class η. Section 3 constructs the fiber integration pushforward and proves current closure. Section 4 presents the modified field equations and the Gauss-law topological charge. Section 5 develops the effective field theory and examines quantum dephasing. Section 6 details the differential-cohomological framework and formulates bulk anomaly-consistency constraints. Section 7 outlines empirical detection limits, and Section 8 summarizes our results.

2 Geometric and Sheaf-Theoretic Framework

Let X denote a 4-dimensional, smooth, oriented, time-orientable pseudo-Riemannian manifold. We define the bulk spacetime Y as the total space of a smooth fiber bundle:

$$K \hookrightarrow Y \xrightarrow{\pi} X \tag{4}$$

where the fiber K is a compact, connected, boundaryless k-dimensional Riemannian manifold ($\partial K = \emptyset$), establishing $\dim_{\mathbb{R}} Y = 4 + k$ [1].

2.1 Categorical Sheaf Formulation

To ensure solidcompatibility with the category of coherent sheaves, we endow Y with the structure of a complex analytic manifold, denoting by $\mathcal{O}_Y$ its structure sheaf of holomorphic functions. Because complex manifolds possess an even real dimension, we explicitly restrict the fiber dimension to even integers $k \in 2\mathbb{N}$ whenever the coherent-sheaf formulation is employed.

Definition 1 (Holomorphic Gauge and Higgs Sheaves). *Let $\mathcal{L} \to Y$ be a holomorphic line bundle over Y. We define:*

1. *The gauge sheaf $\mathcal{E}_P := \mathcal{O}_Y(\mathcal{L}) \in \mathrm{Coh}(Y)$ as the locally free coherent sheaf of holomorphic sections of $\mathcal{L}$ [1]. The underlying smooth complex line bundle carries a $U(1)$ connection, locally represented by gauge potentials $\mathcal{A}$, with globally defined curvature $\mathcal{F} \in \Omega^2_{\mathrm{cl}}(Y)$.*

2. *The bulk Higgs sheaf $\mathcal{H} \in \mathrm{Coh}(Y)$ as a coherent analytic sheaf encoding scalar vacuum geometry on Y [1].*

In the bounded derived category of coherent sheaves $\mathcal{D}^b(Y)$, the interaction between the bulk gauge sector and the Higgs sheaf is represented by an extension class [1].

Definition 2 (Derived Extension Class). *Let $\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H}$ be the derived tensor product of the pulled-back gauge sheaf and the Higgs sheaf on Y. The topological bulk interaction is parameterized by an element of the derived extension group:*

$$\eta \in \mathrm{Hom}_{\mathcal{D}^b(Y)}\big(\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H},\, \mathcal{O}_Y[k+3]\big) \cong \mathrm{Ext}^{k+3}_Y\big(\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H},\, \mathcal{O}_Y\big)\,. \tag{5}$$

2.2 The de Rham Realization Map

To transition from derived sheaf extensions to differential forms on Y, we formulate a characteristic realization postulate.

Geometric Postulate 1 (Characteristic de Rham Realization)**.** *We postulate the existence of a characteristic homomorphism:*

$$\mathfrak{c}_\eta : \mathrm{Ext}^{k+3}_Y\big(\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H},\, \mathcal{O}_Y\big) \longrightarrow H^{k+3}_{\mathrm{dR}}(Y) \tag{6}$$

constructed via the contraction of Atiyah classes $\mathrm{at}(\pi^*\mathcal{E}_P) \in H^1(Y, \Omega^1_Y \otimes \mathrm{End}(\pi^*\mathcal{E}_P))$ *and* $\mathrm{at}(\mathcal{H})$ *with the trace pairing. The extension class* η *determines a de Rham cohomology class* $[\mathcal{J}] = \mathfrak{c}_\eta(\eta) \in H^{k+3}_{\mathrm{dR}}(Y)$*, represented by a smooth, closed differential form* $\mathcal{J} \in \Omega^{k+3}_{\mathrm{cl}}(Y)$ *such that* $d_Y\mathcal{J} = 0$*.*

3 Fiber Integration and the Spacetime Soul Current

The physical current on physical 4D spacetime X is generated by integrating the bulk differential form $\mathcal{J}$ along the fiber K [1].

Definition 3 (Fiber Pushforward Current)**.** *The soul current* J_s *on* X *is defined as the pushforward of* $\mathcal{J}$ *under the bundle projection* π*:*

$$J_s := \pi_*\mathcal{J} = \int_K \mathcal{J} \in \Omega^3(X). \tag{7}$$

Theorem 1 (Conservation of the Soul Current)**.** *Let* $\pi : Y \to X$ *be a smooth fiber bundle with compact, boundaryless fiber* K $(\partial K = \emptyset)$*, and let* $\mathcal{J} \in \Omega^{k+3}_{\mathrm{cl}}(Y)$*. Then the spacetime current* $J_s \in \Omega^3(X)$ *is strictly closed:*

$$d_X J_s = 0. \tag{8}$$

Proof. By the fiberwise Stokes theorem for integration along the fiber of an oriented smooth bundle:

$$d_X(\pi_*\mathcal{J}) = \pi_*(d_Y\mathcal{J}) + (-1)^{\deg(\mathcal{J})-k} \int_{\partial K} \mathcal{J}. \tag{9}$$

Since $d_Y\mathcal{J} = 0$ by Postulate 1 and $\partial K = \emptyset$, the boundary contribution vanishes:

$$d_X J_s = d_X(\pi_*\mathcal{J}) = \pi_*(d_Y\mathcal{J}) = \pi_*(0) = 0. \tag{10}$$

Hence, J_s is a closed differential 3-form on physical spacetime X. □

Proposition 1 (Decoupling in the Four-Dimensional Limit)**.** *Let* $k = 0$*, such that* $Y \cong X$ *and* $\pi = \mathrm{id}_X$*. Under the explicit decoupling condition* $\mathcal{J}_{k=0} \equiv 0$*, the spacetime current satisfies* $J_s \equiv 0$*, identically recovering source-free classical electromagnetism [1].*

4 Modified Electrodynamics and Topological Charge

Let $F \in \Omega^2(X)$ denote the physical electromagnetic field strength on X, and let $*F \in \Omega^2(X)$ be its Hodge dual with respect to the spacetime metric $g_{\mu\nu}$. Because $\deg(*F) = 2$, its exterior derivative $d*F$ is a differential 3-form, matching the degree of $J_s \in \Omega^3(X)$ [1].

Definition 4 (Modified Maxwell Equations)**.** *The extended field equations for the photon field in the presence of bulk topological coupling are:*

$$dF = 0, \tag{11}$$

$$d * F = J_s. \tag{12}$$

Theorem 2 (Gauss-Law Topological Soul Charge). *Let $\Sigma^3 \subset X$ be a spatial 3-chain with boundary $\partial\Sigma^3$. The total enclosed topological soul charge is:*

$$Q_{\text{soul}}(\Sigma^3) = \int_{\Sigma^3} J_s = \int_{\partial\Sigma^3} *F. \tag{13}$$

Proof. Integrating the inhomogeneous equation $d{*}F = J_s$ over Σ^3 and applying Stokes' theorem on X:

$$Q_{\text{soul}}(\Sigma^3) = \int_{\Sigma^3} J_s = \int_{\Sigma^3} d * F = \int_{\partial\Sigma^3} *F. \tag{14}$$

□

Remark 1. *If Σ^3 is a closed 3-cycle ($\partial\Sigma^3 = \emptyset$), Stokes' theorem yields $\int_{\Sigma^3} J_s = \int_{\emptyset} *F = 0$, confirming that $J_s = d * F$ acts as a localized source of differential flux without generating unphysical net topological charges on closed compact manifolds.*

5 Effective Action, Gauge Invariance, and Quantum Dephasing

We construct the low-energy effective field theory on X by coupling the physical electromagnetic gauge potential $A \in \Omega^1(X)$ to the pushforward current $J_s \in \Omega^3(X)$ [1].

Definition 5 (Effective Topological Action). *The action governing physical abelian electrodynamics is:*

$$S_{\text{eff}}[A] = -\frac{1}{4}\int_X F \wedge *F + \int_X A \wedge J_s. \tag{15}$$

Proposition 2 (Gauge Invariance). *$S_{\text{eff}}[A]$ is invariant under the gauge transformation $A \mapsto A + d\lambda$ for any gauge parameter $\lambda \in C_c^\infty(X)$.*

Proof. Under $A \mapsto A + d\lambda$, the variation of the topological interaction term is:

$$\delta_\lambda S_{\text{eff}} = \int_X d\lambda \wedge J_s = \int_X d(\lambda J_s) - \int_X \lambda\, dJ_s. \tag{16}$$

By Stokes' theorem, $\int_X d(\lambda J_s) = \int_{\partial X} \lambda J_s = 0$ for compactly supported λ. Because $dJ_s = 0$ by Theorem 1, the second term vanishes identically:

$$\delta_\lambda S_{\text{eff}} = 0. \tag{17}$$

Hence, the topological source term preserves exact $U(1)$ gauge invariance on X. □

5.1 Quantum Interference Visibility Attenuation

Consider a multi-path quantum interferometer enclosing a spatial 3-chain V_3 bounded by interference paths. The coupling $\int A \wedge J_s$ imparts a geometric phase shift to single-photon states:

$$\Delta\phi_s = \int_{V_3} J_s. \tag{18}$$

Let the compact fiber K be an isotropic k-torus $(S^1)^k$ with characteristic compactification radius $R \sim \Lambda^{-1}$, such that $\mathrm{Vol}(K) \sim \Lambda^{-k}$. Let $\omega(\eta) \in H^3(X)$ denote the characteristic obstruction class induced by the extension η [1]. Assuming stochastic vacuum fluctuations with zero mean $\langle\Delta\phi_s\rangle = 0$ and variance $\langle(\Delta\phi_s)^2\rangle \sim \|\omega(\eta)\|^2\Lambda^{-2k}$, the interference visibility $V = |\langle e^{i\Delta\phi_s}\rangle|$ expands to leading order as:

$$1 - V \approx \frac{1}{2}\langle(\Delta\phi_s)^2\rangle \sim \|\omega(\eta)\|^2\Lambda^{-2k}. \tag{19}$$

We propose this scaling as a phenomenological hypothesis for high-precision quantum optical tests.

6 Differential Cohomology and Anomaly Consistency

To ensure quantization of topological flux, the connection and the current are embedded into Cheeger-Simons differential cohomology $\widehat{H}^\bullet(Y)$ [2].

Geometric Postulate 2 (Differential Cohomological Lift)**.** *We assume that the closed form $\mathcal{J}$ has integral periods:*

$$\mathcal{J} \in \Omega^{k+3}_{\mathbb{Z}}(Y) \tag{20}$$

and therefore admits a smooth differential character lift $\widehat{\eta} \in \widehat{H}^{k+3}(Y)$ with curvature:

$$\mathrm{curv}(\widehat{\eta}) = \mathcal{J}. \tag{21}$$

The fiber integration pushforward in differential cohomology:

$$\pi_* : \widehat{H}^{k+3}(Y) \longrightarrow \widehat{H}^3(X) \tag{22}$$

commutes with the curvature homomorphism [2]:

$$\mathrm{curv}(\pi_*\widehat{\eta}) = \pi_*(\mathrm{curv}(\widehat{\eta})) = \pi_*\mathcal{J} = J_s \in \Omega^3_{\mathrm{cl}}(X). \tag{23}$$

Consequently, the differential character $\pi_*\widehat{\eta} \in \widehat{H}^3(X)$ evaluates to quantized holonomies on integral 2-cycles, with curvature J_s, whose integral over spatial 3-chains determines the corresponding boundary holonomy on $\partial\Sigma^3$ [2].

6.1 Bulk Anomaly-Consistency Conditions

For a complete $(4+k)$-dimensional bulk field theory with chiral matter representations, the gauge and gravitational anomalies are encoded in an anomaly polynomial $I_{k+6}(\mathcal{F}, \mathcal{R})$. Consistency of the dimensional reduction requires the polynomial to satisfy a generalized Green-Schwarz factorization:

$$I_{k+6}(\mathcal{F}, \mathcal{R}) = \sum_a X_a^{(p_a)} \wedge Y_a^{(k+6-p_a)} \tag{24}$$

where X_a and Y_a are invariant polynomials in the curvature forms. Any consistent ultraviolet completion of this framework must specify a bulk chiral fermion spectrum whose one-loop triangle and higher-polygon diagrams cancel against the classical variation of a corresponding Wess-Zumino counterterm on Y.

7 Experimental Constraints and Signatures

The predicted visibility attenuation $1 - V \sim \|\omega(\eta)\|^2\Lambda^{-2k}$ supplies a concrete signature for precision quantum-optical tests:

- **Mach-Zehnder Interferometry**: Ultra-stable optical interferometers operating with phase sensitivities $\delta\phi \sim 10^{-10}\,\mathrm{rad}/\sqrt{\mathrm{Hz}}$ place direct upper bounds on $\|\omega(\eta)\|$ for sub-millimeter compactification scenarios ($k = 2, \Lambda \sim 1\,\mathrm{TeV}$).

- **High-Q Cavity Resonators**: Superconducting cavity QED architectures with long photon dwell times constrain non-Maxwellian phase diffusion, bounding bulk-coupling parameters across macroscopic geometries.

8 Conclusion

We have proposed a geometric extension of Maxwell electrodynamics based on higher-dimensional derived sheaf theory and fiber integration [1]. By parameterizing the bulk interaction via the derived extension class $\eta \in \mathrm{Ext}_Y^{k+3}(\pi^*\mathcal{E}_P \otimes^{\mathbf{L}} \mathcal{H}, \mathcal{O}_Y)$ on a complex analytic fiber bundle $Y \xrightarrow{\pi} X$ with even fiber dimension $k \in 2\mathbb{N}$, mapping η to a closed form $\mathcal{J} \in \Omega_{\mathrm{cl}}^{k+3}(Y)$, and evaluating the pushforward $J_s = \pi_*\mathcal{J} \in \Omega^3(X)$, we established the degree-consistent field equations $dF = 0$ and $d * F = J_s$ [1].

Under the stated geometric postulates, the theory preserves four-dimensional gauge invariance, recovers standard electrodynamics identically under the $k = 0$ decoupling limit, and induces a conserved Gauss-law topological charge $Q_{\mathrm{soul}}(\Sigma^3) = \int_{\partial\Sigma^3} *F$ [1]. Furthermore, we formulated a phenomenological quantum optical prediction: an interference visibility suppression scaling as $1-V \sim \|\omega(\eta)\|^2\Lambda^{-2k}$ [1]. Future investigations will focus on constructing explicit Čech-Dolbeault representatives for the characteristic map $\mathfrak{c}_\eta$, determining complete chiral matter spectra for bulk Green-Schwarz anomaly cancellation, and exploring embeddings into full $SU(2)_L \times U(1)_Y$ electroweak sectors and string compactifications [1, 3].

References

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