Toward Geometric Localization in Kaluza-Klein Gravity: Einstein-Radion Nonlinear Electromagnetic Solitons

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Abstract

We propose a theoretical framework exploring a candidate geometric principle underlying the localization of particle-like states. By building upon Wheeler's concept of geons and introducing non-linear modifications to classical electromagnetism, we investigate a higher-dimensional coupling between extended electromagnetic fields and a dynamic radion field within a compactified spacetime. In this framework, we explore whether some particle-like states conventionally described as elementary excitations may admit a complementary geometric interpretation as localized nonlinear field configurations. Specifically, this framework investigates whether the ADM mass of localized classical configurations can contain a nontrivial contribution arising from dynamical compactification degrees of freedom. This suggests that a localized electromagnetic excitation may manifest coupled to a localized deformation of the higher-dimensional spacetime geometry.

1 Introduction: The Geon Precedent and Localization

The Standard Model of particle physics successfully treats elementary particles as point-like excitations of fundamental quantum fields. However, in classical unified field theories, treating localized field configurations as zero-dimensional points inevitably leads to divergent self-energies. Historically, John Archibald Wheeler's concept of "geons" (gravitational electromagnetic entities) demonstrated that coupled non-linear dynamics of General Relativity and electromagnetism could, in principle, create self-gravitating, localized particle-like entities from pure field energy [1].

Unlike conventional geon models, where gravitational self-interaction provides the primary confinement mechanism, the present framework introduces an additional stabilizing channel: the dynamical deformation of compactification geometry through a radion degree of freedom. The proposed mechanism is therefore not purely gravitational confinement, but a coupled localization phenomenon involving gauge fields, spacetime curvature, and internal geometric moduli. Crucially, rather than claiming that mass emerges purely from geometry, we investigate whether the ADM mass of localized configurations receives a nontrivial contribution from the dynamical compactification modulus, potentially altering the localization and stability properties of the solution. Localization is therefore not attributed to any single sector, but to the balance between non-linear electromagnetic pressure, gravitational backreaction, and radion-mediated compactification energy. The present construction should therefore initially be interpreted as a charged solitonic sector rather than a direct model of electrically neutral elementary particles. Crucially, this framework does not claim that Standard Model particles are classical solitons, nor that quantum properties such as spin and statistics emerge automatically. Instead, it investigates whether localized geometric field configurations can exist as a classical sector of higher-dimensional gravity.

2 Relation to Existing Kaluza-Klein Effective Field Theory

In standard Kaluza-Klein effective field theories derived via dimensional reduction of $(4+n)$-dimensional gravity, internal breathing modes typically manifest as global, spatially homogeneous moduli governing the overall scale of the extra dimensions across the entire four-dimensional universe. By contrast, our framework investigates strictly *spatially localized* solutions of the effective radion scalar. While standard compactifications assume $\phi(x) = \text{const}$ globally, we examine configurations where the radion dynamically deviates from its vacuum expectation value only within the localized electromagnetic core ($r \lesssim r_\phi$) and rapidly decays to the constant vacuum state at spatial infinity. This spatial localization of the scalar modulus is structurally analogous to the trapping of scalar fields in self-gravitating boson stars, with the critical physical distinction that the localized scalar $\phi(r)$ is not an ad-hoc matter field, but represents a localized spatial breathing mode of the internal compactification geometry.

3 Relation to Previous Soliton Models

To contextualize this framework, it is essential to compare it to established soliton families in mathematical physics. Self-gravitating pure field configurations have been extensively studied:

- **Wheeler Geons:** Pure Einstein-Maxwell solutions, which are generally unstable against radial perturbations [1].
- **Boson Stars and Oscillatons:** Self-gravitating localized scalar field configurations where macroscopic mass emerges from scalar energy, though lacking the gauge-geometry coupling proposed here [2].
- **Bartnik-McKinnon Solutions:** Regular, static, self-gravitating solitons in pure Einstein-Yang-Mills theory, existing without Higgs scalars [3].
- **Einstein-Born-Infeld Solitons:** Configurations utilizing non-linear electrodynamics to soften core divergences, yielding finite-energy field distributions [4].
- **Kaluza-Klein Monopoles:** Topological defects arising in 5D gravity, representing static global structures in the extra dimensions [5].
- **Flux Compactifications:** Models stabilizing global extra dimensions via homogeneous p-form fluxes [6].

The distinguishing feature proposed here is the dynamical localization of the compactification modulus around the gauge configuration. Rather than treating the radion as a global background constant, the internal geometry dynamically responds to the local electromagnetic energy density, creating a localized radion-supported core.

4 Relation to Einstein-Maxwell-Dilaton Solitons: Why a Radion is Different

It is crucial to distinguish this framework from standard Einstein-Maxwell-dilaton solitons, Kaluza-Klein bubbles, or flux-supported compactifications. In conventional dilaton models, a scalar field is introduced ad-hoc into the four-dimensional action with arbitrary coupling functions. In contrast, here the scalar ϕ is strictly identified as the physical geometric breathing modulus governing the volume of the extra-dimensional manifold $\mathcal{K}_n$. The radion is interpreted not as an additional matter scalar but as a collective coordinate describing local elastic deformation energy of the compactified manifold. Consequently, the spatial localization of ϕ around the electromagnetic core does not merely represent a scalar soliton, but corresponds directly to a physical, localized spatial deformation of the higher-dimensional compactification geometry itself. To make this structural and ontological distinction explicit, Table 1 contrasts ordinary dilaton models with the proposed radion soliton.

Feature	**Einstein-Maxwell-Dilaton**	**Proposed Radion Soliton**
Field Origin	Fundamental scalar field introduced ad-hoc	Internal metric degree of freedom ($\det g_{ab}$)
Coupling Function $f(\phi)$	Arbitrary choice in effective action	Rigorously fixed by compactification geometry (c_1)
Ontological Status	Additional matter sector	Pure geometric compactification sector
Higher-Dimensional View	No required higher-dimensional interpretation	Direct physical volume deformation of internal KK manifold $\mathcal{K}_n$

Table 1: Systematic comparison separating standard dilaton solitons from the geometric radion framework.

5 Non-Linear Modification of Maxwell's Equations

We consider a $(4+n)$-dimensional bulk action governed by gravity and a non-linear gauge sector. The generalized action is:

$$S_{4+n} = \int d^{4+n}X\sqrt{-G}\left[\frac{M_*^{2+n}}{2}\mathcal{R}^{(4+n)} + \mathcal{L}_{NLED}(F_{MN}F^{MN}) - \Lambda_{bulk}\right] \tag{1}$$

where M_* is the bulk Planck mass and $\mathcal{L}_{NLED}$ represents a non-linear extension of Maxwell's equations. For purely electric configurations where the invariant $(F\tilde{F})^2 = 0$, the Born-Infeld Lagrangian simplifies to $\mathcal{L}_{BI} = \beta^2(1 - \sqrt{1 + F^2/2\beta^2})$. This non-linearity softens the electromagnetic core divergence and provides a mechanism for finite-energy configurations, though global regularity must still be proven for the fully coupled system.

6 Derivation of the Explicit Radion-NLED Coupling

To transition to the four-dimensional effective theory, we assume the factorizable metric ansatz:

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{2\sigma(x)}\gamma_{ab}(y)dy^a dy^b, \tag{2}$$

where the internal compact manifold $\mathcal{K}_n$ has volume $V_\mathcal{K}$ and constant internal curvature $\mathcal{R}_\mathcal{K}$. Under dimensional reduction, the $(4+n)$-dimensional Einstein-Hilbert action yields the four-dimensional scalar-tensor gravity action in the Jordan frame:

$$S_{grav} = \int d^4x\sqrt{-g}\, e^{n\sigma}\left[\frac{M_{\rm Pl}^2}{2}\mathcal{R}^{(4)} + \dots\right]. \tag{3}$$

To eliminate the non-minimal coupling and transition to the four-dimensional Einstein frame, we perform the Weyl conformal rescaling $\tilde{g}_{\mu\nu} = e^{n\sigma}g_{\mu\nu}$ (equivalently, $g_{\mu\nu} = e^{-n\sigma}\tilde{g}_{\mu\nu}$). Under this transformation, the gravitational sector decouples from the breathing mode kinetic term:

$$S_{grav} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^2}{2}\tilde{\mathcal{R}} - \frac{n(n+2)}{4}M_{\rm Pl}^2(\tilde{\partial}\sigma)^2\right]. \tag{4}$$

To cast the internal volume breathing mode into a canonically normalized scalar field with canonical kinetic term $-\frac{1}{2}(\tilde{\partial}\phi)^2$, we define the physical radion field ϕ via:

$$\phi \equiv M_{\rm Pl}\sqrt{\frac{n(n+2)}{2}}\sigma \quad \Longrightarrow \quad \sigma = \sqrt{\frac{2}{n(n+2)}}\frac{\phi}{M_{\rm Pl}}. \tag{5}$$

We now derive the explicit radion-electromagnetic coupling by tracking the scaling weight of the bulk gauge action under dimensional reduction and conformal transformation. Start by reducing the higher-dimensional volume element explicitly:

$$\sqrt{-G} = e^{n\sigma}\sqrt{-g}\sqrt{\gamma}V_{\mathcal{K}}. \tag{6}$$

Assuming a bulk gauge field whose non-vanishing components reside strictly in the four-dimensional spacetime ($F_{\mu\nu}$), the higher-dimensional Maxwell contraction gives:

$$-\frac{1}{4}\sqrt{-G}F_{MN}F^{MN} = -\frac{1}{4}V_{\mathcal{K}}\,e^{n\sigma}\sqrt{-g}\,g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} = V_{\mathcal{K}}\,e^{n\sigma}\sqrt{-g}\,\mathcal{L}_{NLED}(F^2). \tag{7}$$

Crucially, under the subsequent four-dimensional Weyl conformal rescaling to the Einstein frame ($\tilde{g}_{\mu\nu} = e^{n\sigma}g_{\mu\nu}$, where $\sqrt{-g} = e^{-2n\sigma}\sqrt{-\tilde{g}}$ and $g^{\mu\nu} = e^{n\sigma}\tilde{g}^{\mu\nu}$), four-dimensional Maxwell electrodynamics is conformally invariant: the metric contractions exactly cancel ($\sqrt{-g}F^2 = \sqrt{-\tilde{g}}\tilde{F}^2$), generating no Weyl conformal factors. Therefore, the entire radion coupling factor $e^{n\sigma}$ in front of the four-dimensional gauge action originates strictly and exclusively from the higher-dimensional internal volume determinant ($\sqrt{\gamma_{ab}e^{2\sigma}} = e^{n\sigma}\sqrt{\gamma}$). Substituting the canonical radion definition $\sigma = \sqrt{\frac{2}{n(n+2)}}\frac{\phi}{M_{\rm Pl}}$ and adopting the standard downward volume contraction convention ($\phi < 0$), the effective four-dimensional gauge coupling function takes the exact exponential form:

$$f(\phi) = \exp\left(-c_1\frac{\phi}{M_{\rm Pl}}\right), \quad \text{where} \quad c_1 = \sqrt{\frac{2n}{n+2}}. \tag{8}$$

The full 4D Einstein-frame effective action is:

$$S_{eff} = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_{\rm Pl}^2}{2}\tilde{\mathcal{R}} - \frac{1}{2}(\tilde{\partial}\phi)^2 - V_{eff}(\phi) + f(\phi)\mathcal{L}_{NLED}(\tilde{F}^2)\right] \tag{9}$$

The radion effective potential $V_{eff}(\phi)$ balances the internal scalar curvature against a quantized harmonic flux ρ_{flux}, generating a dynamically stable global vacuum state.

7 Effective Stress-Energy Tensor and Four-Dimensional Gravitational Dynamics

To rigorously connect the effective action S_{eff} to the spacetime curvature, we derive the four-dimensional gravitational field equations by varying the action with respect to the metric $\tilde{g}_{\mu\nu}$:

$$G_{\mu\nu} = 8\pi G\left(T_{\mu\nu}^{NLED} + T_{\mu\nu}^{\phi}\right) \tag{10}$$

where $G_{\mu\nu} \equiv \tilde{\mathcal{R}}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{\mathcal{R}}$ is the Einstein tensor. The explicit stress-energy tensor for the dynamical compactification modulus ϕ is given by:

$$T_{\mu\nu}^{\phi} = \partial_\mu\phi\partial_\nu\phi - \tilde{g}_{\mu\nu}\left[\frac{1}{2}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + V_{eff}(\phi)\right] \tag{11}$$

The non-linear electromagnetic sector contributes a stress-energy tensor directly modulated by the radion coupling function $f(\phi)$:

$$T_{\mu\nu}^{NLED} = f(\phi)\left[-2\frac{\partial \mathcal{L}_{NLED}}{\partial \tilde{g}^{\mu\nu}} + \tilde{g}_{\mu\nu}\mathcal{L}_{NLED}\right] \tag{12}$$

For purely electric radial configurations ($E(r) \equiv F_{rt}$), the diagonal components of these stress-energy tensors yield the localized energy density sources governing radial gravitational backreaction:

$$\rho_{NLED} \equiv -T^{t}_{t,NLED} = f(\phi)\left[ED(E) - \mathcal{L}_{NLED}\right], \tag{13}$$

$$\rho_{\phi} \equiv -T^{t}_{t,\phi} = \frac{1}{2}N(\phi')^2 + V_{eff}(\phi). \tag{14}$$

This explicit stress-energy decomposition demonstrates how the local electromagnetic pressure and radion gradients jointly curve the four-dimensional spacetime.

8 The Gauge Field as the Spacetime "Stage": Radial Field Equations

Assuming static spherical symmetry and purely electric configurations, we define the spacetime metric:

$$ds^2 = -N(r)S(r)^2dt^2 + N(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{15}$$

where $N(r) = 1 - \frac{2Gm(r)}{r}$. For the non-linear electrodynamic sector, with radial electric field $E(r) \equiv F_{rt}$, we introduce the electric displacement variable $D(E)$ defined by:

$$D(E) \equiv \frac{\partial \mathcal{L}_{NLED}}{\partial E}. \tag{16}$$

To derive the coupled radial equations from the four-dimensional effective action S_{eff}, we vary the action with respect to the radion scalar ϕ. The Euler-Lagrange equation $\frac{1}{\sqrt{-\tilde{g}}}\partial_\mu(\sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu\phi) - \frac{\partial V_{eff}}{\partial\phi} + \frac{\partial f}{\partial\phi}\mathcal{L}_{NLED} = 0$ in static spherical coordinates yields the exact scalar equation of motion. The complete set of coupled Einstein-Radion-NLED field equations is given by:

$$m' = 4\pi r^2(\rho_{NLED} + \rho_\phi) \tag{17}$$

$$\frac{S'}{S} = 4\pi G r(\rho_{total} + p_{r,total}) \tag{18}$$

$$\left(r^2 S f(\phi) D(E)\right)' = 0 \quad \Longrightarrow \quad r^2 S f(\phi) D(E) = Q \tag{19}$$

$$(r^2 N S \phi')' = r^2 S\left(\frac{\partial V_{eff}}{\partial \phi} - \frac{\partial f}{\partial \phi}\mathcal{L}_{NLED}\right) \tag{20}$$

Equation (20) confirms that for the coupling $+f(\phi)\mathcal{L}_{NLED}$ in the action, the non-linear electromagnetic energy density acts with a negative gradient coupling $-\frac{\partial f}{\partial\phi}\mathcal{L}_{NLED}$ as a localized source for the extra-dimensional volume scalar ϕ. The sign of the effective radion source depends on the electromagnetic Lagrangian convention; throughout this work, the mostly-minus metric signature $(-,+,+,+)$ and the corresponding Born-Infeld sign convention are adopted consistently throughout. As the electromagnetic field propagates, it continuously warps the local compactification geometry.

The conserved quantity Q arises directly as the integration constant of the non-linear Maxwell displacement equation ($r^2 S f(\phi) D(E) = Q$). Whether this classical integration constant can be identified with the quantized electric charge of the Standard Model or represents a topological flux remains an open question.

9 Proposed Geometric Localization Principle

In our framework, the electromagnetic field acts as a direct source for the compactification modulus. Writing the internal radius as

$$R_{\mathcal{K}}(r) = R_0 \exp\left(\frac{\phi(r)}{M_{\text{Pl}}\sqrt{n(n+2)/2}}\right), \tag{21}$$

the compactification radius becomes a dynamical function of the electromagnetic configuration. The logical chain of physical backreaction follows:

$$E_{\text{field}} \longrightarrow E_{\text{geometry}} \longrightarrow M_{\text{ADM}}. \tag{22}$$

To visualize how these physical sectors interact across scales, Figure 1 illustrates the conceptual chain linking higher-dimensional internal compactification geometry to the macroscopic four-dimensional ADM mass.

Higher-Dimensional Geometry ($\mathcal{K}_n$)
$\downarrow$
Dynamical Compactification Radius $R_{\mathcal{K}}(r)$
$\downarrow$
EM Field ($F_{\mu\nu}$) $\longrightarrow$ **NLED Core** ($r \lesssim r_\phi$) $\longrightarrow$ **Radion Deformation** (ϕ)
$\downarrow$
Asymptotic ADM Mass (M_{ADM})

Figure 1: Conceptual hierarchy showing how local non-linear electromagnetic energy warps the internal compactification modulus $\phi(r)$, generating a localized geometric mass contribution to the asymptotic four-dimensional ADM mass.

To explicitly anchor this emergent mass to spacetime geometry without presuming flat-space energy integration rules in a curved background, the asymptotic behavior of the metric lapse function defines the total Arnowitt-Deser-Misner (ADM) mass M_{ADM} of the localized soliton via:

$$N(r) \longrightarrow 1 - \frac{2GM_{\text{ADM}}}{r} \qquad (r \longrightarrow \infty). \tag{23}$$

Equivalently, expressed in terms of the radial mass function $m(r)$ governed by $m'(r) = 4\pi r^2 \rho_{\text{total}}(r)$, the ADM mass is defined by the asymptotic limit:

$$M_{\text{ADM}} = \lim_{r\to\infty} m(r). \tag{24}$$

For the static spherically symmetric mass function gauge adopted here, the asymptotic mass parameter is obtained from the radial integral of the effective energy density:

$$M_{\text{ADM}} = \lim_{r\to\infty} m(r) = 4\pi \int_0^\infty r^2 \rho_{\text{total}}(r) dr, \tag{25}$$

where the total effective energy density $\rho_{\text{total}} = \rho_{NLED} + \rho_\phi$ includes the exact radion contribution:

$$\rho_\phi \equiv \frac{1}{2} N(\phi')^2 + V_{eff}(\phi). \tag{26}$$

This formulation avoids flat-space approximations and explicitly demonstrates that the dynamical compactification modulus ϕ directly contributes to the macroscopic inertial and gravitational mass of the soliton through its kinetic gradient and effective potential energy.

Proposed Geometric Localization Principle

Conjecture (Geometric Localization): *Any finite-energy regular solution of the coupled Einstein-Radion-NLED equations is expected to generate a localized deformation of the compactification manifold. The classical gravitational mass is identified with the ADM energy of the complete localized gauge-geometry configuration.*

10 Dimensionless Formulation for Numerical Integration

To prepare the coupled radial boundary-value problem for direct numerical shooting solvers, we introduce the dimensionless radial coordinate and rescaled field variables:

$$x \equiv M_{\mathrm{Pl}} r, \qquad \varphi(x) \equiv \frac{\phi(r)}{M_{\mathrm{Pl}}}, \qquad \tilde{m}(x) \equiv M_{\mathrm{Pl}} m(r), \qquad \tilde{E}(x) \equiv \frac{E(r)}{M_{\mathrm{Pl}}^2}. \tag{27}$$

Defining the dimensionless control parameters $\lambda \equiv \frac{\beta}{M_{\mathrm{Pl}}^2}$, $\gamma \equiv \frac{Q}{M_{\mathrm{Pl}}}$, and normalized potential $\tilde{V}_{eff}(\varphi) \equiv \frac{V_{eff}(\phi)}{M_{\mathrm{Pl}}^4}$, the four-dimensional Einstein-Radion-NLED differential system reduces to the exact dimensionless first- and second-order ordinary differential equations:

$$\frac{d\tilde{m}}{dx} = 4\pi x^2 \left[f(\varphi)\left(\tilde{E}\tilde{D}(\tilde{E}) - \tilde{\mathcal{L}}_{NLED}\right) + \frac{1}{2} N \left(\frac{d\varphi}{dx}\right)^2 + \tilde{V}_{eff}(\varphi) \right], \tag{28}$$

$$\frac{1}{S}\frac{dS}{dx} = 4\pi x \left[2f(\varphi)\tilde{E}\tilde{D}(\tilde{E}) + N\left(\frac{d\varphi}{dx}\right)^2 \right], \tag{29}$$

$$x^2 S f(\varphi)\tilde{D}(\tilde{E}) = \gamma \quad \Longrightarrow \quad \tilde{E}(x) = \frac{\frac{\gamma}{x^2 S f(\varphi)}}{\sqrt{1 + \left[\frac{\gamma}{\lambda x^2 S f(\varphi)}\right]^2}}, \tag{30}$$

$$\frac{d}{dx}\left(x^2 N S \frac{d\varphi}{dx}\right) = x^2 S \left[\frac{\partial \tilde{V}_{eff}}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\tilde{\mathcal{L}}_{NLED}(\tilde{E}) \right], \tag{31}$$

where $N(x) = 1 - \frac{2\tilde{m}(x)}{x}$ and $\tilde{D}(\tilde{E}) \equiv \frac{\partial \tilde{\mathcal{L}}_{NLED}}{\partial \tilde{E}}$. Here, the explicit algebraic inversion $\tilde{E}(\tilde{D})$ arises directly from the standard Born-Infeld constitutive relation $\tilde{D}(\tilde{E}) = \frac{\tilde{E}}{\sqrt{1-\tilde{E}^2/\lambda}}$, inverted to yield $\tilde{E} = \frac{\tilde{D}}{\sqrt{1+\tilde{D}^2/\lambda}}$ where $\tilde{D}(x) = \frac{\gamma}{x^2 S f(\varphi)}$. The dimensionless Born-Infeld relations below assume the electric-sector convention $\mathcal{L}_{BI} = \beta^2(1 - \sqrt{1 - E^2/\beta^2})$, equivalent up to signature-dependent sign choices. This dimensionless system constitutes the exact, non-linear two-point boundary value problem to be integrated.

11 Reduction to a Concrete Numerical Model and Boundary Conditions

To establish the existence of regular soliton configurations, the coupled system must be formulated as a non-linear two-point boundary value problem solvable via shooting methods.

11.1 Exact Boundary Conditions

At the origin ($r = 0$), regularity requires that the enclosed mass vanishes and all scalar and electromagnetic profiles are smooth with zero radial derivatives, while the electric field takes a finite central saturation value E_c:

$$m(0) = 0, \tag{32}$$
$$\phi'(0) = 0, \tag{33}$$
$$E(0) = E_c, \qquad E'(0) = 0. \tag{34}$$

Crucially, the Born-Infeld sector allows a finite electric field at the origin ($E(0) \to E_c$) despite the conserved electric displacement flux singularity ($D(E) \sim Q/r^2$ as $r \to 0$), providing a possible mechanism for avoiding the Maxwell self-energy divergence, subject to the regularity conditions of the fully coupled Einstein-Radion-NLED system. Strictly, the central saturation value is determined by the local effective displacement $D(0) = Q/[S(0)f(\phi(0))]$, and equals the pure Born-Infeld limit $E_c \to \beta$ only when the local metric and radion coupling factors are appropriately normalized.

At spatial infinity ($r \to \infty$), the configuration must asymptotically match the vacuum spacetime with constant compactification modulus, unit lapse normalization, and vanishing gauge potential:

$$\phi(\infty) = \phi_0, \quad \text{where} \quad V'_{eff}(\phi_0) = 0, \tag{35}$$
$$S(\infty) = 1, \tag{36}$$
$$A_0(\infty) = 0, \tag{37}$$
$$N(r) \longrightarrow 1 - \frac{2GM_{\text{ADM}}}{r}. \tag{38}$$

We define the existence criterion as a non-singular trajectory connecting central shooting parameters $\{\phi(0) = \phi_c, E(0) = E_c, S(0) = S_c\}$ that simultaneously satisfies these asymptotic vacuum boundary conditions at infinity. Here E_c is determined by the shooting procedure or fixed by non-linear core saturation (such as the Born-Infeld maximum field strength $E_c \sim \beta$).

11.2 Model Parameters and Perturbative Limit

We propose specific parameters:

- **Dimensions ($n = 2$):** Defining $c_1 = 1$.
- **Potential:** $V_{eff}(\phi) = Ae^{-a\phi/M_{Pl}} + Be^{-b\phi/M_{Pl}}$, with a stable minimum at ϕ_0 where $m_\phi^2 = V''_{eff}(\phi_0) > 0$.
- **Dimensionless Control Parameters:** $\lambda = \frac{\beta}{M_{Pl}^2}$, $\eta = \frac{m_\phi}{M_{Pl}}$, and $\gamma = \frac{Q}{M_{Pl}}$.

In the weak-gravity, small-radion perturbation limit ($\phi(r) = \phi_0 + \delta\phi(r)$ with $N(r) \approx 1$, $S(r) \approx 1$), equation (20) linearizes to:

$$\nabla^2(\delta\phi) - m_\phi^2\delta\phi \approx -f'(\phi_0)\mathcal{L}_{NLED}(E), \tag{39}$$

which explicitly demonstrates how a localized non-linear electromagnetic core acts as an effective Yukawa source, trapping the radion field within a characteristic radius $r_\phi \sim m_\phi^{-1}$ and generating a finite geometric mass contribution.

12 Mathematical Pathways: Moduli-Space Geometry and Linear Radial Stability Analysis

Solving this highly non-linear boundary-value problem poses a formidable challenge.

12.1 Moduli-Theoretic Interpretation

The solution space may admit a moduli-space interpretation, $\mathfrak{M}_{\mathrm{RSGS}}$. In this picture, the radion acts as the parameter governing geometric deformations of the internal manifold. Continuous curves on the moduli space would correspond to smooth deformations of one geometric state into another, while distinct topological charges would restrict solutions to disconnected components.

12.2 Linear Radial Stability Analysis

Stability against linear radial perturbations is tested by applying spectral analysis to the second-variation Sturm-Liouville operator $-\frac{d^2}{dr_*^2} + \mathbf{U}_{eff}(r_*)$. If the lowest-lying radial eigenfrequency squared ($\omega_0^2 \geq 0$), the configuration contains no exponentially growing radial tachyonic modes, establishing linear radial stability. The present analysis addresses only radial perturbations; full perturbative stability requires decomposition into scalar, vector, and tensor perturbation sectors including non-spherical and angular radion modes.

13 Comparison with Known Solitons

To highlight the novelty and structural stability advantages of the proposed radion-supported non-linear electromagnetic soliton, Table 2 compares this framework against classical self-gravitating soliton models using ‘tabularx‘ to handle wide text.

Model	**Localizing Field**	**Primary Problem**	**Proposed Extension**
Wheeler Geon [1]	EM + gravity	Unstable modes	Radion geometric stabilization
Boson Star [2]	Scalar field	No gauge structure	Electromagnetic localization
Bartnik-McKinnon [3]	Yang-Mills	Non-Abelian	Non-linear Abelian EM
Born-Infeld Particle [4]	Non-linear EM	No gravity backreaction	Full Einstein-radion coupling
EM-Dilaton Solitons	Dilaton + EM	Scalar is ad-hoc	Radion as breathing mode
KK Monopoles [5]	5D gravity	Extra dim. topology	Localized radion response

Table 2: Comparison of classical self-gravitating field configurations.

14 Limitations and Open Problems

- **Existence Unproven:** Globally regular solutions linking the origin to the asymptotic vacuum have not yet been established.
- **Linear vs. Full Stability:** Quasinormal linear radial stability, as well as full multi-sector stability, remain open questions.
- **Asymptotic Coulomb Energy and Charge Interpretation:** For a purely electric localized object in General Relativity, a non-zero conserved charge $Q \neq 0$ implies a long-range Coulomb tail ($E(r) \sim Q/r^2$) whose energy density extends to spatial infinity. Resolving the global energy convergence of such states requires explicit secondary mechanisms: (i) non-linear vacuum screening at large distances, (ii) interpreting Q as a compact internal topological flux rather than an unbounded 4D electric charge, (iii) topological winding quantization, or (iv) a massive gauge sector outside the localized core. A genuinely particle-like finite-energy solution requires either vanishing asymptotic 4D electric flux or a mechanism modifying the infrared gauge behavior. The physically relevant soliton may correspond not to an isolated four-dimensional electric charge, but to a localized higher-dimensional flux tube whose four-dimensional asymptotic gauge charge is screened.

- **Fifth-Force Corrections:** A light radion sector would generally induce fifth-force corrections, requiring decoupled mechanisms or large radion masses.
- **Fermion Interpretation:** Extension to fermionic quantum numbers would require spinorial zero modes or topological quantization to explain spin-1/2 representations.

15 Conclusion

By formulating a concrete Einstein-Radion-NLED system where electromagnetic localization couples directly to deformations of compactification geometry, we investigate whether the ADM mass of localized classical configurations can receive a nontrivial contribution from internal geometric moduli. This framework reduces the problem of geometric localization to a well-defined numerical two-point boundary value problem, and we present these equations as an open challenge to the theoretical physics community.

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