Investigations into an unobserved photonic law of nature: Torsional Boundary States in Warped Compactifications: An Effective APS-Type Index Formula with Nieh-Yan Contributions (Paper IV)

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July 26, 2026

Abstract

This is the first of a new set of three, where we continue to expand upon our initial work. Hercule Poirot had his suspects, we have our hidden variables. Poirot, however, had the distinct advantage of a closed room; our unobserved laws operate in arbitrary dimension without a smoking gun in sight. Following the Topological Emergence Principle, we derive an effective APS-type index formula including torsion-dependent contributions for a classical gauge field confined to the boundary of a warped extra dimension. While previous work in this program established the general Riemann-Cartan holographic framework, this manuscript moves to a specific, realistic compactification geometry: a slice of AdS_5 (a Randall-Sundrum throat) permeated by a background torsional flux. By evaluating the Nieh-Yan invariant explicitly within this warped background, we derive the corresponding torsion-dependent contribution to the bulk spectral asymmetry within the effective framework adopted here. Crucially, we formulate and analyze the generalized equations governing the gauge-field zero mode coupled to this torsional background. Within the effective model, we show that the torsion-dependent contribution can compensate the effective anomaly-inflow contribution, leading to the massless gauge zero-mode localizing on the infrared (IR) boundary brane. This provides an effective geometric mechanism for the Holographic Photon Law: the four-dimensional photon remains structurally intact ($dF = 0$) because its wavefunction localization is topologically protected by the Nieh-Yan invariant of the bulk geometry.

Scope of this Paper

This manuscript constitutes the fourth stage of our ongoing research program into the Topological Emergence Principle. While Papers I–III established the philosophical motivation, the abstract boundary construction, and the macroscopic phenomenological predictions of torsional anomaly inflow, the present work pivots to explicit mathematical physics.

The objective here is strictly calculational. We assume the general holographic framework established previously, but we transition from abstract manifolds to a concrete, testable string-inspired metric. By picking a specific warped compactification geometry, we explicitly calculate the torsional boundary terms that were previously only formally defined. Unless explicitly stated otherwise, the foundational assumptions regarding anomaly cancellation and holographic emergence are carried over from the preceding trilogy.

1 Geometric Setting: The Warped AdS_5 Throat

To evaluate the APS-type index formula including torsion corrections, we define our bulk Y as a five-dimensional manifold with a warped geometry bounded by two 4D branes (the ultraviolet/UV brane at $y = 0$ and the infrared/IR brane at $y = \pi R$). The line element is the standard Randall-Sundrum metric:

$$ds^2 = G_{MN}dx^M dx^N = e^{-2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \tag{1}$$

where $A(y) = ky$ is the warp factor, k is the AdS curvature scale, and $\eta_{\mu\nu}$ is the 4D Minkowski metric.

Let e^a be the vielbeins, where $a, b \in \{0, 1, 2, 3, 5\}$. The non-vanishing components of the Levi-Civita spin connection $\mathring{\omega}^a{}_b$ are entirely determined by the warp factor.

2 Introducing the Torsional Flux

We now extend this geometry to a Riemann-Cartan space (U_5) by introducing a non-vanishing torsion 2-form $T^a = de^a + \omega^a{}_b \wedge e^b$. The full spin connection becomes $\omega^a{}_b = \mathring{\omega}^a{}_b + K^a{}_b$, where $K^a{}_b$ is the contortion tensor.

Motivated by string-theoretic Kalb-Ramond background fluxes, we assume the bulk is permeated by a totally antisymmetric torsion field generated by an H-flux, $H = dB$, such that the torsion components take the form $T_{abc} = H_{abc}$. For the sake of boundary state localization, we assume a background axial torsional vector $S_M = \epsilon_{MNPQR}T^{NPQR}$ directed entirely along the extra dimension: $S_\mu = 0, S_y \neq 0$.

3 Explicit Calculation of the Nieh-Yan Invariant

The Nieh-Yan 4-form N_{NY} is the best-known independent torsional invariant in Riemann-Cartan geometry. It is given by:

$$N_{\text{NY}} = T^a \wedge T_a - R_{ab} \wedge e^a \wedge e^b = d(e^a \wedge T_a), \tag{2}$$

where R_{ab} is the full Riemann-Cartan curvature 2-form.

In our warped throat with axial torsion S_y, we evaluate this exterior derivative using Stokes' Theorem. The induced 4D vielbein scaling on a slice $y =$ const takes the form $e^{\hat{a}} = e^{-ky}\hat{e}^{\hat{a}}$, where $\hat{a} \in \{0, 1, 2, 3\}$. Under the assumed warped torsional background, the induced vierbein on the IR brane contributes an overall warp factor $e^{-2k\pi R}$, yielding:

$$\int_Y N_{\text{NY}} = \int_{\partial Y} e^a \wedge T_a = e^{-2k\pi R}\int_{X_{\text{IR}}} (\hat{e}^{\hat{a}} \wedge T_{\hat{a}}) - \int_{X_{\text{UV}}} (\hat{e}^{\hat{a}} \wedge T_{\hat{a}}). \tag{3}$$

In this background, the warp factor exponentially weights the IR boundary contribution relative to the UV boundary contribution. Within this warped geometry, the exponential weighting of the boundary contribution exponentially enhances the effective Nieh–Yan contribution associated with the IR boundary.

4 The Effective APS-Type Index Formula

We now apply this specific background to the Atiyah-Patodi-Singer index theorem for the bulk Dirac operator D_Y. In effective Riemann-Cartan descriptions, torsion-dependent corrections can be represented by additional geometric invariants supplementing the conventional characteristic classes.

Although index theorems for Dirac operators on manifolds with torsion have been studied in various settings, no universal modification of the classical Atiyah–Patodi–Singer theorem exists. Motivated by these observations, throughout this work we employ the following effective index functional as a phenomenological description of the torsion-dependent contributions within the present framework. The following expression should therefore be regarded as the defining equation of the effective model developed in this paper:

$$\boxed{\mathrm{ind}_{\mathrm{eff}}(D_Y) = \int_Y \left(\hat{A}(\mathring{R}) \wedge \mathrm{ch}(E) + \alpha N_{\mathrm{NY}} \right) - \frac{\eta(D_X) + h}{2}} \tag{4}$$

We emphasize that $\mathrm{ind}_{\mathrm{eff}}$ is introduced as an effective phenomenological functional motivated by known torsional invariants and anomaly-inflow arguments. It is not claimed to represent a universal extension of the classical Atiyah–Patodi–Singer index theorem. Rather, it provides a mathematically consistent framework for exploring the consequences of torsion within the Topological Emergence Principle.

Here α denotes an effective coupling whose normalization depends upon the chosen regularization and conventions. Since the Nieh-Yan contribution is known to depend on the regularization prescription in quantum field theory, the normalization of this effective term should be interpreted within the chosen regularization scheme.

Following the Holographic Vacuum Ansatz introduced in Paper I, we impose the condition $\mathrm{ind}_{\mathrm{eff}}(D_Y) = 0$. Setting $\mathrm{ind}_{\mathrm{eff}}(D_Y) = 0$ within the effective model yields the balancing condition:

$$\frac{\eta(D_X) + h}{2} = \int_Y \hat{A}(\mathring{R}) \wedge \mathrm{ch}(E) + \alpha e^{-2k\pi R} \int_{X_{\mathrm{IR}}} (\hat{e}^{\hat{a}} \wedge T_{\hat{a}}). \tag{5}$$

Within the present framework this implies: the spectral asymmetry of the 4D boundary ($\eta(D_X)$) is constrained by the torsional flux pushed onto the IR brane by the warp factor.

5 Gauge Zero-Mode Localization

To show that this topology protects the 4D photon, we evaluate the gauge field equations of motion. The action for the gauge field coupled to the bulk torsion is:

$$S_{\mathrm{bulk}} = \int d^5x \sqrt{-G} \left(-\frac{1}{4} F_{MN} F^{MN} + \frac{c}{2} S^M A^N \tilde{F}_{MN} \right), \tag{6}$$

where the second term represents the torsional Chern-Simons-like coupling identified in Paper III. We adopt the effective torsional Chern-Simons coupling motivated by the anomaly inflow analysis of Paper III.

Working in the $A_y = 0$ gauge and performing a variational derivative of S_{bulk} with respect to A_μ, the 5D field equation reduces to:

$$\partial_N \left(\sqrt{-G} G^{NP} G^{\mu Q} F_{PQ} \right) + c S_y \partial_y A^\mu = 0. \tag{7}$$

Substituting the warped metric determinant $\sqrt{-G} = e^{-4ky}$ and metric components $G^{\mu\nu} = e^{2ky} \eta^{\mu\nu}$ yields:

$$\eta^{\nu\rho} \partial_\nu F_{\rho\mu} + \partial_y \left(e^{-2ky} \partial_y A_\mu \right) + c S_y \partial_y A_\mu = 0. \tag{8}$$

We apply a Kaluza-Klein expansion $A_\mu(x, y) = \sum_n A_\mu^{(n)}(x) f_n(y)$, where the 4D field satisfies the massive wave equation $\eta^{\nu\rho} \partial_\nu F_{\rho\mu}^{(n)} = -m_n^2 e^{2ky} A_\mu^{(n)}$. After imposing the warped background, the Kaluza–Klein decomposition, and the assumptions of the effective torsional model, the resulting Sturm-Liouville equation governing the profile functions takes the form:

$$-\partial_y \left(e^{-2ky} \partial_y f_n \right) + m_n^2 e^{2ky} f_n - c S_y \partial_y f_n = 0. \tag{9}$$

For the massless zero-mode ($m_0 = 0$), the equation simplifies drastically to:

$$\partial_y \left(e^{-2ky} \partial_y f_0 \right) + cS_y \partial_y f_0 = 0. \tag{10}$$

The presence of the topological torsion term S_y, which is bound to the Nieh-Yan invariant via Equation (5), interacts with the boundary geometry. Under the approximation of slowly varying torsion, one obtains the approximate solution for the zero-mode:

$$f_0(y) \approx N_0 \exp\left(\int dy \left(\frac{cS_y}{e^{-2ky}} \right) \right). \tag{11}$$

The exponential should be interpreted as the leading-order WKB solution obtained after neglecting derivatives of the slowly varying torsion background. A complete spectral analysis of the Sturm-Liouville operator lies beyond the scope of the present work. Because the warp factor exponentially enhances the IR contribution within the effective index functional, the model predicts exponential localization for the class of slowly varying torsion backgrounds considered here near $y = \pi R$.

Interpretational Remark

This calculation provides an explicit realization of the Holographic Photon Law proposed in the preceding papers. The massless 4D photon ($dF = 0$) exists perfectly on our spacetime brane not by accident, but because the Nieh-Yan topological invariant of the bulk Riemann-Cartan geometry provides a geometric explanation for the localization of its wavefunction within the effective framework.

Epistemological Status

Warped AdS geometry	Established
Nieh-Yan 4-form calculation	Established
KK Decomposition	Established
Torsional zero-mode localization	Derived within the present effective model
Connection to Holographic Law	Interpreted within the effective framework

6 Conclusion

By moving from abstract manifolds to a concrete warped AdS_5 compactification, we have explicitly calculated the interactions between the Nieh-Yan invariant, the APS index theorem, and gauge field localization. Within the effective framework developed here, we have constructed a torsion-dependent APS-type description that exponentially enhances the effective boundary contribution associated with the anomaly inflow, naturally leading to the massless gauge zero-mode localizing there.

Having developed the geometric origin of the boundary state, the final required step for this research program is to extract quantitative empirical bounds. Paper V will leverage this localized boundary action to compare our predicted torsional vacuum birefringence against recent high-precision Cosmic Microwave Background (CMB) polarization data.

References

[1] L. Randall, R. Sundrum, *A Large Mass Hierarchy from a Small Extra Dimension*, Phys. Rev. Lett. 83, 3370-3373 (1999).

[2] H.T. Nieh, M.L. Yan, *Quantized Dirac Field in Curved Riemann-Cartan Background. I. Symmetry Properties, Green's Function*, Ann. Phys. 138, 237-259 (1982).

[3] Y.N. Obukhov, *The Nieh–Yan invariant and fermions in Riemann–Cartan geometry*, Phys. Lett. A 90, 13–16 (1982).

[4] J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. 284, 681-699 (1989).

[5] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42, 5-84 (2006).

[6] A. Pomarol, *Gauge Bosons in a Five-Dimensional Theory with Localized Gravity*, Phys. Lett. B 486, 153-157 (2000).

[7] M.F. Atiyah, V.K. Patodi, I.M. Singer, *Spectral asymmetry and Riemannian Geometry. I*, Math. Proc. Cambridge Philos. Soc. 77 (1975).

[8] O. Chandía, J. Zanelli, *Topological Invariants, Instantons and Chiral Anomaly on Spaces with Torsion*, Phys. Rev. D 55, 7580-7585 (1997).

[9] E.W. Mielke, *Anomalies and Gravity*, AIP Conference Proceedings 857, 246 (2006).

[10] I.L. Shapiro, *Physical aspects of the space-time torsion*, Phys. Rep. 357, 113-213 (2002).

[11] T. Gherghetta, A. Pomarol, *Bulk fields and supersymmetry in a slice of AdS*, Nuclear Physics B 586, 141-162 (2000).

[12] Y. Grossman, M. Neubert, *Neutrino masses and mixings in non-factorizable geometry*, Phys. Lett. B 474, 361-371 (2000).