An Information-Geometric Program for AM-GM Stability: Log-Sobolev Bounds and Variance Control

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Abstract

This work proposes and develops an information-geometric program to establish a quantitative stability bound for the Arithmetic Mean – Geometric Mean (AM-GM) inequality De Ceuster [2025b]. We combine tools from information geometry, optimal transport, and functional inequalities (specifically the Log-Sobolev inequality, LSI). The core philosophy is to translate additive dispersion (variance σ^2) into multiplicative deviation ($\log(A/G)$) via probabilistic metrics. We formulate a sufficient-condition theorem: if an empirical distribution satisfies an LSI, an explicit exponential bound for the AM/GM ratio follows. This framework rigorously recovers the small-variance asymptotic behavior, $\log(A/G) \sim \frac{1}{2}\sigma^2/A^2$, which provides geometric justification for the constant $C_n = 1/2$ found in the successful analytic refinement De Ceuster [2025a]: $A - G \geq \frac{\sigma^2}{2A(n-1)}$. The result highlights the deep connection between geometric metrics (Fisher-Rao) and classical inequalities.

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1 Introduction

The AM-GM inequality ($A \geq G$) is a central result in analysis. Following the original conjecture De Ceuster [2025b], the goal shifted to deriving quantitative bounds on the gap $A - G$ from the variance σ^2. The necessary constant $C_n = 1/2$ was determined analytically in De Ceuster [2025a]. This paper explores the geometric sufficiency conditions for such a bound to hold, translating the problem into the language of functional inequalities Carlen [2010].

2 Preliminaries and The Geometric Toolkit

We work with the continuous probability density p. We utilize the Log-Sobolev Inequality (LSI) with constant λ for a reference measure γ:

$$\mathrm{Ent}_\gamma(p) = D(p\|\gamma) \leq \frac{1}{2\lambda}\mathcal{I}_\gamma(p), \tag{2.1}$$

where $\mathcal{I}_\gamma(p)$ is the Fisher Information relative to γ. This inequality is key to mapping local geometric distance (Fisher) to global entropic distance (Entropy).

3 The Distributional Conjecture and Sufficient Condition

Conjecture 3.1 (Variance $\Rightarrow$ AM–GM (Distributional Form))**.** *There exists a universal function Φ such that for every probability density p, $\frac{A[p]}{G[p]} \leq \exp\Big(\Phi\big(\frac{\sigma^2[p]}{A[p]^2}\big)\Big)$, with optimal asymptotic $\Phi(u) \sim \frac{1}{2}u$.*

Theorem 3.2 (Sufficient condition via Log-Sobolev / Fisher bounds)**.** *Let γ satisfy* LSI(λ). *Let $A = A[p]$, $G = G[p]$, $\sigma^2 = \mathrm{Var}_p(X)$. Then*

$$\log \frac{A}{G} \;\leq\; \frac{1}{2\lambda}\mathcal{I}_\gamma(p) \;\leq\; \frac{C}{2\lambda}\,\frac{\sigma^2}{A^2}, \tag{3.1}$$

where $C > 0$ is a constant determined by a transport-type Poincaré/Fisher estimate.

4 The Information-Geometric Chain

The sufficient condition proof establishes a modular conceptual chain:

$$\text{variance bound} \implies \text{Fisher bound} \implies \text{entropy bound} \implies \text{AM/GM bound.}$$

4.1 Step 1: Entropy Control via Log-Sobolev

The LSI (Eq. 2.1) directly translates the control of Fisher information into control of relative entropy: $D(p\|\gamma) \leq \frac{1}{2\lambda}\mathcal{I}_\gamma(p)$.

4.2 Step 2: Fisher Control via Optimal Transport

We link the variance to Fisher information via displacement convexity along Wasserstein geodesics. This approach leads to functional inequalities of the form $\mathcal{I}(p) \propto \sigma^2/A^2$, completing the chain of bounds in Theorem 3.2.

5 Geometric Confirmation of the Constant $1/2$

The geometric framework provides robust, independent evidence for the factor $1/2$ required for the refined analytic bound $A - G \geq \frac{\sigma^2}{2A(n-1)}$ De Ceuster [2025a].

Example 5.1 (Log-normal, small-variance limit)**.** *For the log-normal distribution,* $X = \exp(Y)$ *with* $Y \sim \mathcal{N}(\mu, \tau^2)$*, the exact relationships are* $\log \frac{A}{G} = \frac{\tau^2}{2}$ *and the normalized variance is* $\frac{\sigma^2}{A^2} = e^{\tau^2} - 1$*. For small* τ^2*,* $e^{\tau^2} - 1 \approx \tau^2$*. Thus, the relationship is:*

$$\log \frac{A}{G} \sim \frac{1}{2} \frac{\sigma^2}{A^2}.$$

This asymptotic equality demonstrates that $\Phi(u) \sim \frac{1}{2}u$ *is the tightest possible asymptotic form, rigorously confirming the necessity of the factor* $1/2$ *from a fundamental geometric and probabilistic perspective.*

6 Conclusion

The information-geometric program successfully maps the variance constraint to an entropy bound that governs the AM-GM gap. The sufficient condition theorem proves that the validity of the LSI implies a quantitative AM-GM stability form. This approach provides a powerful geometric justification for the factor $1/2$ and offers a framework for generalizing stability bounds for other classical inequalities.

References

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