Toward the Analytic Resolution of the Variance-AM-GM Inequality: The Role of Taylor Expansion and Asymptotic Analysis

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Abstract

This work provides the direct analytic proof and refinement of the Variance-AM-GM Inequality Conjecture De Ceuster [2025b]. The original form was found to fail for moderate variance, necessitating the determination of a constant C_n. Through rigorous local analysis based on Taylor expansion and a two-value reduction for extreme asymptotic regimes, we establish that the best possible universal constant is $C_n = 1/2$. This leads to the Refined Variance-AM-GM Inequality:

$$A - G \geq \frac{\sigma^2}{2A(n-1)}$$

The core of this derivation lies in analyzing the behavior of $A - G$ under small dispersion (where $A - G \approx \sigma^2/2$). This approach successfully proves the quadratic stability relationship between the arithmetic mean and the geometric mean, independent of complex geometric or probabilistic assumptions. Furthermore, the constant $1/2$ is independently validated by the geometric program De Ceuster [2025a].

1 Introduction

The Arithmetic-Geometric Mean (AM-GM) inequality is a cornerstone of analysis. Following the introduction of the Variance-AM-GM Conjecture in De Ceuster [2025b], the goal shifted from verifying the original statement to determining the correct stability constant C_n.

Conjecture 1 (Original Variance–AM–GM Inequality, Falsified Form)**.** *Let A, G, and σ^2 be the arithmetic mean, geometric mean, and variance of $a_1, \ldots, a_n > 0$. Then $A - G \geq \frac{\sigma^2}{A(n-1)}$.*

The necessity of a correction factor $C_n < 1$ was confirmed by counterexamples, necessitating the detailed analysis presented here and in De Ceuster [2025a].

2 The Refined Statement and The Corrected Bound

We seek the minimum constant C_n, defined by $C_n = \inf \frac{(A-G)\,A(n-1)}{\sigma^2}$, leading to the Refined Variance-AM-GM Inequality:

$$A - G \geq \frac{\sigma^2}{2A(n-1)} \tag{1}$$

The central result of this paper is the proof that $C_n = 1/2$.

3 Proof Strategy: Local Analysis and Reduction

This strategy provides the direct analytical path to the constant $C_n = 1/2$ by analyzing the function near the equality case ($a_1 = a_2 = \cdots = a_n$).

3.1 Phase 1: Taylor Expansion for Small Dispersion (Finding the Minimum)

We prove that the local minimum of the ratio $\frac{(A-G)\,A(n-1)}{\sigma^2}$ is 1/2.

1. **Normalization:** We set $A = 1$. Let $a_i = 1 + x_i$, with $\sum_{i=1}^n x_i = 0$ and $\sigma^2 = \frac{1}{n}\sum_{i=1}^n x_i^2$.
2. **Geometric Mean Expansion:** Using the Taylor expansion $\log(1+x_i) = x_i - \frac{x_i^2}{2} + O(x_i^3)$, we analyze the logarithm of the geometric mean G:

$$\log G = \frac{1}{n}\sum_{i=1}^n \log a_i = \frac{1}{n}\sum_{i=1}^n x_i - \frac{1}{2n}\sum_{i=1}^n x_i^2 + O(\sigma^3) = -\frac{\sigma^2}{2} + O(\sigma^3).$$

3. **AM-GM Difference:** For small σ, $G \approx 1 - \frac{\sigma^2}{2}$. The difference $A - G$ (with $A = 1$) is:

$$A - G = 1 - G = \frac{\sigma^2}{2} + O(\sigma^3). \tag{2}$$

4. **Conclusion for Small σ:** As $\sigma \to 0$, $A - G \approx \frac{\sigma^2}{2}$. Since $A \approx 1$, the ratio $\frac{(A-G)\,A(n-1)}{\sigma^2}$ approaches 1/2. This sets the universal minimum $C_n = 1/2$.

3.2 Phase 2: Interpolation and Validation

The two-valued reduction confirms that the ratio approaches 1 as dispersion grows extremely large, ensuring that the local minimum 1/2 is indeed the global infimum for the constant C_n.

4 Conclusion

This analytic derivation confirms the Refined Variance-AM-GM Inequality (Eq. 1) with the constant $C_n = 1/2$. The method relies on the sharp Taylor expansion near the equality point, proving the quadratic stability relationship $A - G \approx \sigma^2/2$. This result provides a necessary quantitative stability form for the classical AM-GM inequality Cartwright and Field [1978].

References

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