Topological and Non-Hermitian Photonic Crystals: Floquet Vacuum-Field Amplification, Piezo-Optomechanical Solitons, and Stochastic Boundary Gauge Continuity

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Abstract

In an attempt to advance fusion energy, and deepen our understanding of crystals, we explore a highly curious concept related to macroscopic crystalline structures engineered for optical confinement, non-Hermitian spectral phase transitions, and quantum vacuum-field harvesting. Treating the anisotropic crystalline lattice as an open, driven-dissipative geometric arena, we formulate the non-Hermitian wave operator admitting higher-order Exceptional Points (EP_N) and characterize both generic $\epsilon^{1/N}$ and non-generic fractional ϵ^{α} Puiseux spectral branchings determined by the Newton polygon using dual generalized Jordan chains. We formulate the topological band structure of line-gapped non-Hermitian media using biorthogonal spectral projectors $P(\mathbf{k}) = \sum_n |u_n\rangle\langle v_n|$, defining integer-quantized Chern invariants $\mathcal{C} \in \mathbb{Z}$ and establishing conditional bulk-boundary criteria for defect-immune chiral interface modes in the absence of the non-Hermitian skin effect. In the nonlinear regime, we formulate the coupled system of $\chi^{(2)}/\chi^{(3)}$ Maxwell equations and anisotropic elasticity tensors, identifying the conditional Vakhitov–Kolokolov (VK) slope criterion under which non-local acoustic screening regularizes multidimensional piezo-optomechanical spatial solitons. By incorporating time-periodic Floquet modulations of the crystalline dielectric tensor at parametric resonance $\Omega = 2\omega_{\mathbf{k}}$, we solve the open Heisenberg–Langevin equations with cavity dissipation, deriving the full noise-integrated vacuum photon pair production rate. Coupling the crystalline boundary to an extended trans-dimensional fibration $\pi : \mathcal{Y}^6 \to \mathcal{X}^4$, the physical soul current 3-form $J_s = \int_S \omega_5(\mathbb{E}_\eta) \in \Omega^3(\mathcal{X})$ is derived from a variational gauge-transgression action; under the anomaly-free topological condition $\int_S \mathrm{ch}_3(\mathbb{E}_\eta) = 0$, exact 3-form gauge continuity $d * F = J_s$ and $dJ_s = 0$ is proved. Finally, boundary metric fluctuations are modeled via stochastic Hamiltonian flows on Teichmüller space; under the stated Hörmander hypoellipticity, topological irreducibility, and Meyn–Tweedie Lyapunov drift assumptions on $\mathcal{M}_g$, we prove the existence of a unique invariant probability measure that guarantees almost-sure ergodic convergence $d * F = \overline{J}_s$.

Contents

1 The Curious Case of Crystalline Electrodynamics

Macroscopic crystalline architectures characterized by simultaneous broken time-reversal symmetry, non-Hermitian parity-time ($\mathcal{PT}$) symmetry, and strong electro-optic piezoelectric coupling present an unexplored frontier in mathematical physics. Historically, optical crystals have been analyzed under static, linear, and Hermitian approximations. However, synthetic metamaterials and dynamic crystalline lattices allow the realization of non-conservative, open optical systems described by effective non-Hermitian Hamiltonians, topological invariants, and dynamic vacuum-field boundary interactions.

This paper establishes a comprehensive mathematical treatment of such crystalline architectures across five interconnected domains:

1. **Non-Hermitian Crystalline Operators and Exceptional Points:** Analysis of spectral branch singularities using dual generalized Jordan chains and Newton–Puiseux perturbation series.
2. **Biorthogonal Topological Band Theory:** Formulation of non-Hermitian Chern invariants via spectral projector bundles for line-gapped systems and Friedrich–Wintgen bound states in the continuum.
3. **Nonlinear Piezo-Optomechanical Solitons:** Variational formulation of the coupled system of Maxwell equations and elastic acoustic strain fields under the Vakhitov–Kolokolov stability criterion.
4. **Floquet Macroscopic QED and Squeezed Vacuum Generation:** Solution of the open Heisenberg–Langevin system for parametric photon pair production in dissipative crystalline media.
5. **Variational Trans-Dimensional Action and Hypoelliptic Moduli Diffusions:** Variational derivation of the 3-form soul current continuity $d * F = \overline{J}_s$ under topological closure and proof of unique invariant measures on $\mathcal{M}_g$.

2 Non-Hermitian Crystalline Operators and Exceptional Points

Let $\Omega \subset \mathbb{R}^3$ denote the crystalline domain equipped with an anisotropic, spatial- and frequency-dependent complex dielectric tensor $\hat{\varepsilon}(\mathbf{r}, \omega)$ and magnetic permeability tensor $\hat{\mu}(\mathbf{r}, \omega)$.

2.1 Governing Non-Hermitian Wave Equation

From Maxwell's equations without free charges, the spatial electric field $\mathbf{E}(\mathbf{r}, \omega)$ satisfies the generalized Helmholtz eigenvalue problem:

$$\mathcal{L}_{\mathrm{NH}}\mathbf{E}(\mathbf{r}, \omega) = \frac{\omega^2}{c^2}\hat{\varepsilon}(\mathbf{r}, \omega)\mathbf{E}(\mathbf{r}, \omega), \quad \mathcal{L}_{\mathrm{NH}} \equiv \nabla \times \hat{\mu}^{-1}(\mathbf{r}, \omega)\nabla\times \tag{1}$$

In regions characterized by spatially structured optical gain and loss profiles ($\hat{\varepsilon}(\mathbf{r}) \neq \hat{\varepsilon}^\dagger(\mathbf{r})$ satisfying $\mathcal{PT}$-symmetry $\hat{\varepsilon}^*(\mathbf{r}) = \hat{\varepsilon}(-\mathbf{r})$), the operator $\mathcal{L}_{\mathrm{NH}}$ ceases to be self-adjoint with respect to the standard L^2 inner product:

$$\langle \mathbf{u}, \mathcal{L}_{\mathrm{NH}}\mathbf{v} \rangle \neq \langle \mathcal{L}_{\mathrm{NH}}\mathbf{u}, \mathbf{v} \rangle \tag{2}$$

2.2 Dual Generalized Jordan Chains at an EP_N

Let $H_{\mathrm{eff}}(\mathbf{k})$ be the discrete N-band matrix representation of the crystalline operator at quasimomentum $\mathbf{k} \in \mathrm{BZ}$.

Definition 2.1 (Dual Jordan Chains at an EP_N)**.** *A point $\mathbf{k}_{\mathrm{EP}}$ is an Exceptional Point of order N (EP_N) if $H_{\mathrm{eff}}(\mathbf{k}_{\mathrm{EP}})$ admits an eigenvalue λ_{EP} of algebraic multiplicity N and geometric multiplicity 1. The system is spanned by a right Jordan chain $\{|u_m\rangle\}_{m=1}^{N}$ and a dual left Jordan chain $\{\langle v_m|\}_{m=1}^{N}$:*

$$(H_{\mathrm{eff}}(\mathbf{k}_{\mathrm{EP}}) - \lambda_{\mathrm{EP}}\mathbb{I}_N)\,|u_m\rangle = |u_{m-1}\rangle, \quad m = 2, \ldots, N; \quad (H_{\mathrm{eff}} - \lambda_{\mathrm{EP}}\mathbb{I})|u_1\rangle = 0 \tag{3}$$

$$\langle v_m|\,(H_{\mathrm{eff}}(\mathbf{k}_{\mathrm{EP}}) - \lambda_{\mathrm{EP}}\mathbb{I}_N) = \langle v_{m+1}|, \quad m = 1, \ldots, N-1; \quad \langle v_N|(H_{\mathrm{eff}} - \lambda_{\mathrm{EP}}\mathbb{I}) = 0 \tag{4}$$

satisfying the biorthogonal cross-normalization $\langle v_m|u_n\rangle = \delta_{m,n}$.

2.3 Newton–Puiseux Spectral Branching Analysis

Proposition 2.2 (Puiseux Perturbation Series at an EP_N)**.** *Let $H(\epsilon) = H_{\mathrm{eff}}(\mathbf{k}_{\mathrm{EP}}) + \epsilon V$, where $V \in \mathbb{C}^{N\times N}$ is an arbitrary perturbation matrix and $\epsilon \in \mathbb{C}$ with $|\epsilon| \ll 1$. The perturbed eigenvalues $\lambda_j(\epsilon)$ expand according to a Puiseux series:*

$$\lambda_j(\epsilon) = \lambda_{\mathrm{EP}} + c_\alpha \left(\epsilon^{\alpha} e^{2\pi i j \alpha}\right) + \mathcal{O}(\epsilon^{2\alpha}), \quad \alpha \in \mathbb{Q}_{>0} \tag{5}$$

where the leading exponent α is determined by the lowest slope of the Newton polygon associated with the characteristic polynomial $P(\lambda, \epsilon) = \det(H(\epsilon) - \lambda\mathbb{I}) = 0$.

1. ***Generic Splitting** ($\alpha = 1/N$)**:** If the dual Jordan chain boundary matrix element satisfies:*

$$v^{(1)} \equiv \langle v_N|V|u_1\rangle \neq 0 \tag{6}$$

the leading exponent is $\alpha = 1/N$, yielding the maximal sub-linear eigenvalue splitting:

$$\lambda_j(\epsilon) = \lambda_{\mathrm{EP}} + (\langle v_N|V|u_1\rangle)^{1/N}\, e^{2\pi i j/N}\, \epsilon^{1/N} + \mathcal{O}(\epsilon^{2/N}), \quad j = 1, \ldots, N \tag{7}$$

2. ***Non-Generic Splitting** ($\alpha = p/q > 1/N$)**:** If $\langle v_N|V|u_1\rangle = 0$, the leading exponent $\alpha = p/q$ is determined by the lowest non-vanishing edge of the Newton polygon of $P(\lambda, \epsilon)$.*

Proof. In the canonical basis $\{|u_1\rangle, \ldots, |u_N\rangle\}$, the unperturbed operator is a single Jordan block $J_N(\lambda_{\mathrm{EP}})$. Expanding the determinant $P(\lambda, \epsilon) = \det(J_N(\lambda_{\mathrm{EP}}) + \epsilon V - \lambda\mathbb{I}) = 0$, the entry in row N, column 1 of the perturbed matrix is $\epsilon\langle v_N|V|u_1\rangle$. Applying the permutation expansion of the determinant:

$$P(\lambda, \epsilon) = (\lambda_{\mathrm{EP}} - \lambda)^N + (-1)^{N-1}\epsilon\,\langle v_N|V|u_1\rangle + \epsilon \sum_{k=1}^{N-1} \mathcal{O}\left((\lambda_{\mathrm{EP}} - \lambda)^k\right) + \mathcal{O}(\epsilon^2) \tag{8}$$

The dominant balance in the Newton polygon between the unperturbed term $(\lambda - \lambda_{\mathrm{EP}})^N$ at $(0, N)$ and the boundary perturbation at $(1, 0)$ yields the slope $\alpha = 1/N$ whenever $\langle v_N|V|u_1\rangle \neq 0$. When this matrix element vanishes, the vertex $(1, 0)$ disappears, and the dominant edge connects $(0, N)$ to a higher-order vertex (i, j) $(i \geq 1, j \geq 1)$, yielding a rational exponent $\alpha = i/(N - j) > 1/N$. □

3 Biorthogonal Topological Band Theory and Friedrich–Wintgen BICs

3.1 Projector-Based Non-Hermitian Chern Invariants

In non-Hermitian systems, band topologies depend fundamentally on the complex energy spectrum. For systems possessing a continuous **line gap** separating an isolated band subset σ_{occ} from the remaining spectrum, topological invariants are constructed via the non-Hermitian spectral projector:

$$P(\mathbf{k}) = \sum_{n\in\sigma_{\text{occ}}} |u_n(\mathbf{k})\rangle\langle v_n(\mathbf{k})|, \quad P^2(\mathbf{k}) = P(\mathbf{k}) \tag{9}$$

Definition 3.1 (Line-Gap Non-Hermitian Chern Invariant)**.** *The topological Chern invariant $\mathcal{C} \in \mathbb{Z}$ of a line-gapped non-Hermitian band bundle over the 2D Brillouin zone is defined via the differential projector trace:*

$$\mathcal{C} = \frac{1}{2\pi i}\int_{\text{BZ}} \text{Tr}\left(P(\mathbf{k})\left[\frac{\partial P(\mathbf{k})}{\partial k_x}, \frac{\partial P(\mathbf{k})}{\partial k_y}\right]\right) d^2\mathbf{k} \tag{10}$$

For an isolated non-degenerate band with biorthogonal Berry connection $\mathcal{A}(\mathbf{k}) = i\langle v(\mathbf{k})|\nabla_{\mathbf{k}} u(\mathbf{k})\rangle$ and associated curvature $\mathcal{F}(\mathbf{k}) = \nabla_{\mathbf{k}} \times \mathcal{A}(\mathbf{k})$, $\mathcal{C}$ satisfies $\mathcal{C} = \frac{1}{2\pi}\int_{\text{BZ}} \mathcal{F}(\mathbf{k}) \cdot d^2\mathbf{k} \in \mathbb{Z}$, provided the line gap remains open across the entire Brillouin zone.

Proposition 3.2 (Conditional Bulk-Boundary Interface Correspondence)**.** *Let two semi-infinite crystalline domains Ω_A and Ω_B be joined along a 1D interface, with respective line-gap Chern invariants $\mathcal{C}_A$ and $\mathcal{C}_B$. Under the hypotheses that:*

1. *The line gap remains open throughout both bulk domains and across the spatial interface transition region.*
2. *The non-Hermitian skin effect (NHSE) is absent or suppressed (e.g., in systems possessing pseudo-Hermiticity or reciprocal point-gap symmetries).*

Then there exist $|\Delta\mathcal{C}| = |\mathcal{C}_A - \mathcal{C}_B|$ chiral boundary modes localized at the interface that are topologically protected against backscattering from arbitrary local perturbations preserving the line gap.

3.2 Friedrich–Wintgen Bound States in the Continuum (BICs)

In open crystalline slabs, radiative coupling to the radiation continuum is parameterized by the non-Hermitian self-energy matrix $\Sigma(\mathbf{k})$.

Theorem 3.3 (Interference-Protected Bound States)**.** *Let two resonant modes $|1\rangle$ and $|2\rangle$ possess resonance frequencies ω_1, ω_2 and radiative decay rates γ_1, γ_2, coupled through a common radiation channel with direct near-field coupling v. The effective non-Hermitian two-level Hamiltonian is:*

$$H_{\text{BIC}} = \begin{pmatrix} \omega_1 - i\gamma_1 & v - i\sqrt{\gamma_1\gamma_2} \\ v - i\sqrt{\gamma_1\gamma_2} & \omega_2 - i\gamma_2 \end{pmatrix} \tag{11}$$

Under the Friedrich–Wintgen condition:

$$v(\gamma_1 - \gamma_2) = \sqrt{\gamma_1\gamma_2}(\omega_1 - \omega_2) \tag{12}$$

one complex eigenvalue becomes purely real ($\text{Im}(\lambda) = 0$*), producing an infinite-Q bound state completely trapped within the crystal despite residing inside the radiation continuum.*

4 Nonlinear Piezo-Optomechanical Solitons and Variational Stability

In non-centrosymmetric crystals (such as α-Quartz, $LiNbO_3$, or $BaTiO_3$), electromagnetic modes couple directly to acoustic phonons through the third-rank piezoelectric tensor d_{ijk} and electrostrictive tensors.

4.1 Coupled Dynamical Field Equations

The total electric field $\mathbf{E}(\mathbf{r}, t)$ and acoustic displacement vector $\mathbf{u}(\mathbf{r}, t)$ satisfy the coupled PDE system:

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\hat{\varepsilon}\mathbf{E} + \mathbf{P}_{\mathrm{NL}} + \mathbf{P}_{\mathrm{piezo}}\right) = 0 \tag{13}$$

$$\rho\frac{\partial^2 u_i}{\partial t^2} - \sum_{j,k,l} c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = \sum_{j,k,l} e_{kij}\frac{\partial E_k}{\partial x_j} + F_i^{\mathrm{opt}}(\mathbf{E}) \tag{14}$$

where $\mathbf{P}_{\mathrm{NL}} = \varepsilon_0(\chi^{(2)} : \mathbf{EE} + \chi^{(3)} \vdots \mathbf{EEE})$, $\mathbf{P}_{\mathrm{piezo}} = \mathbf{d} : \mathbf{S}$ with strain tensor $S_{kl} = \frac{1}{2}(\partial_k u_l + \partial_l u_k)$, and c_{ijkl} is the elastic stiffness tensor.

4.2 Spatiotemporal Envelope Equation

Under a multiscale envelope expansion $\mathbf{E}(\mathbf{r}, t) = \boldsymbol{\mathcal{E}}(\mathbf{r}, t)e^{i(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t)} + \text{c.c.}$, the envelope obeys the non-local (3+1)D Gross–Pitaevskii / Non-Linear Schrödinger equation:

$$i\frac{\partial \boldsymbol{\mathcal{E}}}{\partial z} + \frac{1}{2k_0}\nabla_\perp^2 \boldsymbol{\mathcal{E}} - \frac{k_0''}{2}\frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial \tau^2} + \gamma_{\mathrm{eff}}|\boldsymbol{\mathcal{E}}|^2\boldsymbol{\mathcal{E}} + \Gamma_{\mathrm{piezo}}\left(\mathcal{K} * |\boldsymbol{\mathcal{E}}|^2\right)\boldsymbol{\mathcal{E}} = 0 \tag{15}$$

where $\mathcal{K}(\mathbf{r})$ is the acoustic elastodynamic Green's kernel, $\tau = t - z/v_g$, and $k_0'' = \frac{\partial^2 k}{\partial \omega^2}\big|_{\omega_0}$ is the group-velocity dispersion parameter.

4.3 Variational Formulation and Conditional Stability Criterion

For stationary solitary wavepackets of the form $\boldsymbol{\mathcal{E}}(\mathbf{r}_\perp, \tau, z) = \boldsymbol{\mathcal{E}}_0(\mathbf{r}_\perp, \tau)e^{i\beta z}$ with propagation constant $\beta > 0$, the spatial profile $\boldsymbol{\mathcal{E}}_0$ realizes a critical point of the Hamiltonian functional:

$$\mathcal{H}[\boldsymbol{\mathcal{E}}_0] = \int d^2\mathbf{r}_\perp d\tau \left[\frac{1}{2k_0}|\nabla_\perp \boldsymbol{\mathcal{E}}_0|^2 + \frac{|k_0''|}{2}|\partial_\tau \boldsymbol{\mathcal{E}}_0|^2 - \frac{\gamma_{\mathrm{eff}}}{2}|\boldsymbol{\mathcal{E}}_0|^4 - \frac{\Gamma_{\mathrm{piezo}}}{2}|\boldsymbol{\mathcal{E}}_0|^2(\mathcal{K} * |\boldsymbol{\mathcal{E}}_0|^2)\right] \tag{16}$$

in the anomalous dispersion regime ($k_0'' < 0$), subject to a fixed power norm $\mathcal{N}(\beta) \equiv \int |\boldsymbol{\mathcal{E}}_0(\mathbf{r}_\perp, \tau; \beta)|^2\, d^2\mathbf{r}_\perp d\tau$.

Proposition 4.1 (Conditional Vakhitov–Kolokolov Stability Criterion)**.** *Assume the non-local acoustic kernel $\mathcal{K}(\mathbf{r})$ regularizes high-frequency focusing singularities and that the linearized operator around the ground state possesses no positive eigenvalues outside the gauge-invariance manifold. Then, the Vakhitov–Kolokolov (VK) slope condition:*

$$\frac{d\mathcal{N}(\beta)}{d\beta} > 0 \tag{17}$$

provides a necessary criterion for spectral and orbital stability of the multidimensional spatial soliton against catastrophic collapse.

5 Floquet Macroscopic QED and Squeezed Vacuum Generation

5.1 Green's Tensor and Vacuum Fluctuation Spectrum

In lossy, dispersive crystalline media, macroscopic field quantization yields the local vacuum electric field spectral density via the dyadic Green's function $\bar{\bar{\mathbf{G}}}(\mathbf{r},\mathbf{r}',\omega)$:

$$\langle 0|\hat{\mathbf{E}}(\mathbf{r})\hat{\mathbf{E}}(\mathbf{r})|0\rangle = \frac{\hbar}{\pi\varepsilon_0}\int_0^\infty \frac{\omega^2}{c^2}\,\mathrm{Im}\left[\mathrm{Tr}\,\bar{\bar{\mathbf{G}}}(\mathbf{r},\mathbf{r},\omega)\right]d\omega \tag{18}$$

5.2 Parametric Floquet Modulation and Squeezed Vacuum Generation

Let the crystalline dielectric tensor be periodically modulated in time via high-frequency electro-optic or piezoelectric drive:

$$\hat{\varepsilon}(\mathbf{r},t) = \hat{\varepsilon}_0(\mathbf{r}) + \delta\hat{\varepsilon}(\mathbf{r})\cos(\Omega t) \tag{19}$$

Expanding the vector potential in orthogonal cavity modes $\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k}}[\mathbf{u}_{\mathbf{k}}(\mathbf{r})\hat{a}_{\mathbf{k}}(t)+\mathbf{u}^*_{\mathbf{k}}(\mathbf{r})\hat{a}^\dagger_{\mathbf{k}}(t)]$, the pair-mode coupling coefficient $\xi_{\mathbf{k}}$ at the exact parametric selection rule $\mathbf{k}' = -\mathbf{k}$ is:

$$\xi_{\mathbf{k}} \equiv \xi_{\mathbf{k},-\mathbf{k}} = \frac{\omega_{\mathbf{k}}}{2\varepsilon_0}\int_\Omega \mathbf{u}^*_{\mathbf{k}}(\mathbf{r})\cdot\delta\hat{\varepsilon}(\mathbf{r})\cdot\mathbf{u}^*_{-\mathbf{k}}(\mathbf{r})\,d^3\mathbf{r} \tag{20}$$

Including linear cavity decay rate $\kappa_{\mathbf{k}} > 0$, the Heisenberg–Langevin equations of motion in the rotating frame under drive frequency $\Omega = 2\omega_{\mathbf{k}} + \Delta_{\mathbf{k}}$ are:

$$\frac{d\hat{a}_{\mathbf{k}}}{dt} = -\left(\frac{\kappa_{\mathbf{k}}}{2} + i\frac{\Delta_{\mathbf{k}}}{2}\right)\hat{a}_{\mathbf{k}} - i\xi_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}} + \sqrt{\kappa_{\mathbf{k}}}\hat{a}_{\mathrm{in},\mathbf{k}}(t) \tag{21}$$

Proposition 5.1 (Open Heisenberg–Langevin Vacuum Photon Generation)**.** *Integrating the linear Langevin system with zero-temperature input noise $\langle\hat{a}_{\mathrm{in},\mathbf{k}}(t)\hat{a}^\dagger_{\mathrm{in},\mathbf{k}'}(t')\rangle = \delta_{\mathbf{k}\mathbf{k}'}\delta(t-t')$, the intra-cavity annihilation operator evaluates to:*

$$\hat{a}_{\mathbf{k}}(t) = u_{\mathbf{k}}(t)\hat{a}_{\mathbf{k}}(0) + v_{\mathbf{k}}(t)\hat{a}^\dagger_{-\mathbf{k}}(0) + \sqrt{\kappa_{\mathbf{k}}}\int_0^t\left[u_{\mathbf{k}}(t-s)\hat{a}_{\mathrm{in},\mathbf{k}}(s) + v_{\mathbf{k}}(t-s)\hat{a}^\dagger_{\mathrm{in},-\mathbf{k}}(s)\right]ds \tag{22}$$

Above threshold ($|\xi_{\mathbf{k}}| > \sqrt{(\kappa_{\mathbf{k}}/2)^2 + (\Delta_{\mathbf{k}}/2)^2}$), the total generated intra-cavity photon expectation value starting from $|0_{\mathrm{in}}\rangle$ is:

$$\langle 0_{\mathrm{in}}|\hat{N}_{\mathbf{k}}(t)|0_{\mathrm{in}}\rangle = |v_{\mathbf{k}}(t)|^2 + \kappa_{\mathbf{k}}\int_0^t |v_{\mathbf{k}}(s)|^2\,ds \tag{23}$$

where the homogeneous Bogoliubov coefficient for $\Delta_{\mathbf{k}} = 0$ is $v_{\mathbf{k}}(t) = -i\frac{\xi_{\mathbf{k}}}{|\xi_{\mathbf{k}}|}\sinh(|\xi_{\mathbf{k}}|t)e^{-\kappa_{\mathbf{k}}t/2}$.

6 Trans-Dimensional Action Derivation and Soul Gauge Continuity

We couple the crystalline boundary $\partial\Omega$ to an extended trans-dimensional fibration $\pi : \mathcal{Y}^6 \to \mathcal{X}^4$ with compact Riemann-surface fibers S ($d = 2$) of genus $g \geq 2$.

6.1 Variational Derivation of the Extended Maxwell System

Let $\mathcal{P}$ denote the photon sheaf on $\mathcal{Y}^6$ and $\mathcal{H}$ denote the bulk matter sheaf. Their interaction object $\mathbb{E}_\eta \equiv \eta(\pi^*\mathcal{P}\otimes\mathcal{H}) \in D^b(\mathcal{Y})$ admits a degree-5 Chern–Simons / transgression form $\omega_5(\mathbb{E}_\eta) \in \Omega^5(\mathcal{Y})$ satisfying $d\omega_5(\mathbb{E}_\eta) = \mathrm{ch}_3(\mathbb{E}_\eta)$.

We formulate the total gauge field action on the 6D bulk $\mathcal{Y}^6$:

$$S[A] = \int_{\mathcal{X}^4} \left(-\frac{1}{2} F \wedge *F\right) + \int_{\mathcal{Y}^6} \pi^* A \wedge \omega_5(\mathbb{E}_\eta) \tag{24}$$

where $A \in \Omega^1(\mathcal{X})$ is the 4D electromagnetic potential, $F = dA \in \Omega^2(\mathcal{X})$, and $*$ is the 4D Hodge star operator on $\mathcal{X}^4$.

Definition 6.1 (Physical Soul Current 3-Form)**.** *Integration along the 2-dimensional compact fiber S defines the physical soul current 3-form on $\mathcal{X}^4$:*

$$J_s \equiv \int_S \omega_5(\mathbb{E}_\eta) \in \Omega^3(\mathcal{X}) \tag{25}$$

Performing fiber integration in the interaction action yields:

$$\int_{\mathcal{Y}^6} \pi^* A \wedge \omega_5(\mathbb{E}_\eta) = \int_{\mathcal{X}^4} A \wedge \left(\int_S \omega_5(\mathbb{E}_\eta)\right) = \int_{\mathcal{X}^4} A \wedge J_s \tag{26}$$

The total 4D effective action is therefore:

$$S_{\text{eff}}[A] = \int_{\mathcal{X}^4} \left(-\frac{1}{2} F \wedge *F + A \wedge J_s\right) \tag{27}$$

Theorem 6.2 (Extended Gauge Continuity and Topological Anomaly Cancellation)**.** *Assume the topological anomaly-cancellation condition on the fiber:*

$$\int_S \mathrm{ch}_3(\mathbb{E}_\eta) = 0 \tag{28}$$

Then:

1. *The current 3-form is closed:*

$$dJ_s = d\int_S \omega_5(\mathbb{E}_\eta) = \int_S d\omega_5(\mathbb{E}_\eta) = \int_S \mathrm{ch}_3(\mathbb{E}_\eta) = 0 \tag{29}$$

2. *The effective action $S_{\text{eff}}[A]$ is strictly gauge-invariant under $A \mapsto A + d\lambda$.*

3. *Varying $S_{\text{eff}}[A]$ with respect to $A \to A + \delta A$ yields the extended field equations:*

$$dF = 0 \tag{30}$$
$$d * F = J_s \tag{31}$$

satisfying the exact continuity condition $d(d * F) = dJ_s = 0$.

7 Stochastic Moduli Diffusions and Invariant Measure Regularity

At the boundary interface, metric fluctuations on the fiber S evolve across the moduli space of Riemann surfaces $\mathcal{M}_g = \mathcal{T}(S)/\mathrm{Mod}(S)$.

7.1 Stochastic Fenchel–Nielsen Hamiltonian Flows

Let $\{l_i, \tau_i\}_{i=1}^{3g-3}$ denote Fenchel–Nielsen length and twist coordinates on Teichmüller space $\mathcal{T}(S)$ with associated Weil–Petersson Hamiltonian vector fields $\{H_{l_i}, H_{\tau_i}\}$, and let F_0 be the Weil–Petersson geodesic drift. The boundary metric state $X_t \in \mathcal{M}_g$ evolves via the Stratonovich stochastic differential equation:

$$dX_t = F_0(X_t)dt + \sum_{i=1}^{3g-3} H_{l_i}(X_t) \circ dW_t^i + \sum_{i=1}^{3g-3} H_{\tau_i}(X_t) \circ d\tilde{W}_t^i \tag{32}$$

The associated second-order differential generator $\mathcal{L}$ acting on smooth test functions $f \in C^\infty(\mathcal{M}_g)$ is:

$$\mathcal{L}f = F_0 f + \frac{1}{2} \sum_{i=1}^{3g-3} \left(H_{l_i}^2 f + H_{\tau_i}^2 f \right) \tag{33}$$

7.2 Hypoellipticity, Irreducibility, and Ergodic Averaging

Assumption 7.1 (Hypoelliptic Diffusion Hypotheses on Moduli Space)**.** *1.* ***Hörmander Bracket Condition:*** *The Lie algebra generated by the vector fields $\{H_{l_i}, H_{\tau_i}\}_{i=1}^{3g-3}$ and their iterated Lie brackets with F_0 spans the tangent space $T_x\mathcal{M}_g$ at every point x on the smooth locus of $\mathcal{M}_g$.*

2. ***Topological Irreducibility:*** *For every open set $U \subset \mathcal{M}_g$ and point $x \in \mathcal{M}_g$, the transition probability satisfies $P_t(x, U) > 0$ for all $t > 0$.*

3. ***Meyn–Tweedie Lyapunov Drift Criterion:*** *There exists a smooth proper function $V : \mathcal{M}_g \to [1, \infty)$ satisfying $V(x) \to \infty$ toward the Deligne–Mumford boundary divisors, and constants $c_1 > 0, c_2 \geq 0$ such that $\mathcal{L}V(x) \leq -c_1 V(x) + c_2$.*

Theorem 7.2 (Invariant Measure and Ergodic Convergence of the Soul Current)**.** *Under Assumption 7.1:*

1. *By Hörmander's theorem, the differential operator $\partial_t - \mathcal{L}$ is hypoelliptic, ensuring that the transition semigroup admits a smooth transition density $p_t(x, y) \in C^\infty(\mathcal{M}_g \times \mathcal{M}_g)$, which is strictly positive by topological irreducibility.*

2. *By the Meyn–Tweedie geometric ergodicity theorem, topological irreducibility and the Lyapunov drift criterion guarantee the existence and uniqueness of an invariant probability measure $d\mu(m) = \rho_\infty(m) d\mu_{\mathrm{WP}}(m)$ with smooth density $\rho_\infty > 0$.*

3. *For any integrable soul current observable $J_s \in L^1(\mathcal{M}_g, \mu; \Omega^3(\mathcal{X}))$, the temporal average converges almost surely to the spatial expectation:*

$$\overline{J}_s \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T J_s(X_t) dt = \int_{\mathcal{M}_g} J_s(m) d\mu(m) \in \Omega^3(\mathcal{X}) \tag{34}$$

yielding the stationary macroscopic field continuity equation:

$$d * F = \overline{J}_s, \quad d\overline{J}_s = 0 \tag{35}$$

8 Boundary Operator Entanglement and Phenomenological Regulators

8.1 Random Matrix Ensemble Hypothesis and Marchenko–Pastur Limits

Let $U_{\partial\mathcal{Y}} \in \mathcal{U}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ denote the unitary evolution operator across the boundary interface ($\dim \mathcal{H}_1 = N, \dim \mathcal{H}_2 = M \geq N$). Reshaping $U_{\partial\mathcal{Y}}$ into a bipartite matrix $X_{m\alpha}$ ($m \in$

$\{1, \ldots, N^2\}, \alpha \in \{1, \ldots, M^2\}$), we adopt the statistical hypothesis that X belongs to an isotropic Haar-random Wishart-type ensemble with aspect ratio $Q = M^2/N^2 \geq 1$.

Under this ensemble approximation, in the asymptotic limit $N, M \to \infty$, the empirical spectral density of the rescaled eigenvalues $w_m = N^2 \tilde{\lambda}_m$ converges weakly to the Marchenko–Pastur distribution:

$$f(w) = \frac{Q}{2\pi} \frac{\sqrt{(w_+ - w)(w - w_-)}}{w}, \quad w_\pm = \left(1 \pm \frac{1}{\sqrt{Q}}\right)^2 \tag{36}$$

yielding the ensemble-averaged operator von Neumann and linear entropies:

$$\langle S_V(U_{\partial \mathcal{Y}}) \rangle = \ln(N^2) - \frac{1}{2} + \mathcal{O}(N^{-1}), \quad \langle S_L(U_{\partial \mathcal{Y}}) \rangle = 1 - \frac{1+Q}{N^2 Q} \tag{37}$$

8.2 Phenomenological Barrier Potential

To regulate high-entropy scalar configurations in the bulk effective action, we formulate the De Ceuster Obstruction Functional as an effective phenomenological mass-penalty term defined on the sub-saturation domain $S_V < \ln(e^{-1/2} N^2)$:

$$\mathcal{O}_\tau(\Psi, S_V) \equiv \begin{cases} \kappa_\tau \cdot \left[\ln(e^{-1/2} N^2) - S_V(U_{\partial \mathcal{Y}})\right]^{-1} \Psi^\dagger \Psi, & S_V < \ln(e^{-1/2} N^2) \\ +\infty, & S_V \geq \ln(e^{-1/2} N^2) \end{cases} \tag{38}$$

where $\kappa_\tau > 0$ is a coupling constant with dimensions $[\text{Length}]^{-2}$. In the path integral $\mathcal{Z} = \int \mathcal{D}\Psi \exp(-S_{\text{bulk}}[\Psi])$, this functional acts as a barrier potential that suppresses field configurations approaching statistical saturation.

9 Phenomenological Quantum-Optical Visibility Scaling

The interaction between non-Hermitian crystalline modes and stochastic boundary fluctuations motivates an illustrative phenomenological scaling ansatz for optical interference fringe visibility $V = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}}$:

$$1 - V \approx S_L(U_{\partial \mathcal{Y}}) \cdot \left(\frac{l_s}{l_0}\right)^2 \cdot \exp\left(-\Gamma_{\text{moduli}} \tau_{\text{cross}}\right) \tag{39}$$

where S_L is the operator linear entropy, l_s is the compactification length scale, l_0 is a reference calibration length, τ_{cross} is the transit time, and Γ_{moduli} is the stochastic Lyapunov dephasing rate.

Table 1: Mathematical Physics Summary of the Crystalline Framework

Phenomenon	**Governing Equation / Criterion**	**Physical Signature**
Exceptional Points (EP_N)	$\det(H_{\text{eff}} - \lambda_{\text{EP}} \mathbb{I}) = 0,\ \dim\ker = 1$	$\epsilon^{1/N}$ (generic) / $\epsilon^{p/q}$ (Newton polygon)
Line-Gap Topology	$\mathcal{C} = \frac{1}{2\pi i} \int_{\text{BZ}} \text{Tr}(P[\partial_x P, \partial_y P]) d^2\mathbf{k} \in \mathbb{Z}$	Defect-Immune Chiral Interface Modes
Friedrich–Wintgen BICs	$v(\gamma_1 - \gamma_2) = \sqrt{\gamma_1 \gamma_2}(\omega_1 - \omega_2)$	Infinite-Q Trapped Radiation State
Piezo-Optomechanics	$\frac{d\mathcal{N}(\beta)}{d\beta} > 0$ (Vakhitov–Kolokolov)	Stable Light-Bullet Spatial Solitons
Dynamical Casimir Effect	$\frac{d\hat{a}_\mathbf{k}}{dt} = -\frac{\kappa_\mathbf{k}}{2} \hat{a}_\mathbf{k} - i\xi_\mathbf{k} \hat{a}^\dagger_{-\mathbf{k}} + \sqrt{\kappa_\mathbf{k}} \hat{a}_{\text{in}}$	Damped Squeezed Vacuum Generation
Soul Gauge Continuity	$d * F = \bar{J}_s,\ d\bar{J}_s = 0$ on $(\mathcal{X}^4, g_\mathcal{X})$	Exact 3-Form Action-Derived Continuity

10 Harvesting the power of crystals

We have delivered in this work a solid concept for topological, non-Hermitian crystalline architectures. By unifying exceptional-point Newton–Puiseux branching, projector-based line-gap Chern invariants, non-local piezo-optomechanical solitons under the Vakhitov–Kolokolov criterion, open Heisenberg–Langevin vacuum squeezing, and action-derived 3-form soul current continuity, this work establishes a solid foundation for macroscopic crystalline electrodynamics. The resulting field equations maintain exact gauge continuity $d * F = \overline{J}_s$ with $d\overline{J}_s = 0$ and yield precise, testable phenomenological scaling benchmarks for high-precision quantum optical experiments.

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