

Explicit Calabi-Yau Cohomology, Chiral Spectra, and $SO(10) \subset E_8 \times E_8$ Heterotic Embeddings of Topological Photonic Sheaves

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Abstract

We formulate a Grand Unified embedding framework and cohomological representation for higher-dimensional photonic continuity over a ten-dimensional spacetime product manifold $Y = X \times K$, where X is four-dimensional physical spacetime and K is a compact Calabi-Yau threefold ($k = 6$, $\dim_{\mathbb{R}} Y = 10$) [1]. The standard embedding of the spin connection in an $SU(3)$ factor of the visible E_8 breaks the gauge symmetry to E_6 ; subsequent symmetry breaking to $SO(10)$ and the Standard Model group $SU(3)_C \times SU(2)_L \times U(1)_Y$ is modeled via discrete Wilson lines and bulk Higgsing [1]. The unbroken electromagnetic direction is identified with an adjoint line subbundle $\mathcal{L}_\gamma^{\text{ad}} \subset \text{ad}(P_{E_8})$ stabilized by the electric charge generator $Q = T^3 + Y_W$ [1].

Addressing the vanishing of harmonic 1-forms on strict Calabi-Yau manifolds ($b_1(K) = 0$), we utilize the canonical horizontal-vertical graded decomposition $\Omega^9(Y) = \bigoplus_{p+q=9} \Omega^p(X) \otimes \Omega^q(K)$ [1]. We formulate a postulated closed, integral differential 9-form representative $\mathcal{J} \in \Omega_{\text{cl},\mathbb{Z}}^9(Y)$ whose leading vertical component $\mathcal{J}_{(3,6)} = \omega_3(X) \wedge \nu_K$ (where $\nu_K \in \Omega_{\text{cl},\mathbb{Z}}^6(K)$) is a normalized closed integral representative of the fundamental cohomology class of K with $\int_K \nu_K = 1$, and $\omega_3(X) \in \Omega_{\text{cl},\mathbb{Z}}^3(X)$ is a closed spacetime 3-form) guarantees an exact, non-trivial fiber pushforward soul current $J_s = \pi_* \mathcal{J} = \int_K \mathcal{J}_{(3,6)} = \omega_3(X) \in \Omega^3(X)$ satisfying $d_X J_s = 0$ [1].

We examine the embedding of this structure into $E_8 \times E_8$ heterotic compactifications under consistent adjoint trace conventions ($\text{Tr} \equiv \text{Tr}_{\mathbf{248}}$ for E_8 and $\text{tr} \equiv \text{tr}_{\text{vector}}$ for Lorentz curvature), where the ten-dimensional anomaly 12-form factorizes as $I_{12}(\mathcal{F}, \mathcal{R}) = X_4 \wedge X_8$. The classical variation of the Green-Schwarz counterterm $S_{\text{GS}} \propto \int_Y B_2 \wedge X_8$ cancels the factorized one-loop anomaly, providing a consistent ten-dimensional background for the proposed anomaly-inflow interpretation of J_s [1].

Furthermore, on the Tian-Yau complete intersection Calabi-Yau threefold $\tilde{K} \subset \mathbb{P}^3 \times \mathbb{P}^3$ of bidegrees $(3, 0), (0, 3), (1, 1)$ with Euler characteristic $\chi(\tilde{K}) = -18$ and freely acting symmetry $G \cong \mathbb{Z}_3$, the smooth quotient $K = \tilde{K}/\mathbb{Z}_3$ has $\chi(K) = -6$. Under the standard embedding $\mathcal{V}_F \cong TK$, the Atiyah-Singer index theorem yields a net chiral generation index $\text{ind}(\mathcal{D}_K) = n_{27} - n_{\overline{27}} = \frac{1}{2}|\chi(K)| = 3$, supplying the topological family count of the Standard Model modulo non-chiral vector-like pairs. Finally, we evaluate the phenomenological signatures across precision optical interferometry ($1 - V \approx \frac{1}{2}\langle(\Delta\phi_s)^2\rangle \sim \|\omega(\eta)\|^2 \Lambda^{-12}$), loop-level shifts in the effective weak mixing angle $\sin^2 \theta_W^{\text{eff}}(q^2)$, and the muon anomalous magnetic moment $(g - 2)_\mu$ [1].

1 Introduction

Unifying the Standard Model within higher-dimensional supergravity and superstring compactifications requires addressing both local gauge anomalies and the geometric origin of low-energy symmetries [3, 4]. In preceding foundational work, we proposed the *Photon Soul Continuity* framework, demonstrating that four-dimensional abelian electrodynamics can be extended by

a closed 3-form current $J_s = \pi_* \mathcal{J} \in \Omega^3(X)$ arising from the fiber pushforward of a higher-dimensional bulk characteristic form on a manifold Y [1].

While the degree consistency and Gauss-law charge conservation of the modified field equations $dF = 0$ and $d * F = J_s + eJ_{\text{em}}$ were established in low-energy effective settings [1], embedding this mechanism into a Grand Unified heterotic framework requires addressing three specific structural questions:

1. **Grand Unified Symmetry Breaking:** Physical electromagnetism must be formulated as an unbroken line subbundle surviving spontaneous symmetry breaking from Grand Unified gauge groups such as $SO(10)$ and E_6 embedded in heterotic $E_8 \times E_8$ string compactifications, with the intermediate stages explicitly identified as compactification and Wilson-line breaking steps [3].
2. **Cohomological Representation and Vertical Decomposition:** On strict Calabi-Yau threefolds ($k = 6$, $\dim_{\mathbb{R}} Y = 10$), the first Betti number vanishes ($b_1(K) = 0$), precluding the existence of non-trivial harmonic 1-forms on the internal manifold. The bulk differential 9-form $\mathcal{J}$ must therefore be formulated using horizontal-vertical bidegrees (p, q) on the product space $Y = X \times K$ to ensure that fiber integration $\pi_* : \Omega^9(Y) \rightarrow \Omega^3(X)$ produces a non-vanishing, closed 3-form current on physical spacetime [1].
3. **Anomaly Matching and the Tian-Yau Index:** The ten-dimensional gauge and gravitational anomalies must factorize according to the Green-Schwarz mechanism under consistent trace conventions, while the zero modes of the internal Dirac operator must yield a net chiral index of three on an explicitly specified quotient Calabi-Yau threefold [1].

In this paper, we address these challenges. We embed the gauge sector into an $E_8 \times E_8$ heterotic compactification over the product spacetime $Y = X \times K$, where $K = \tilde{K}/\mathbb{Z}_3$ is the Tian-Yau quotient Calabi-Yau threefold with $\chi(K) = -6$. We formulate the unbroken photonic projection sheaf within the adjoint bundle $\text{ad}(P_{E_8})$, propose the closed 9-form representative $\mathcal{J}_{(3,6)} = \omega_3(X) \wedge \nu_K$, verify Green-Schwarz anomaly matching, and compute the resulting precision phenomenological signatures across atomic parity violation, precision optical interferometry, and muon $(g - 2)$ [1].

2 Grand Unified Gauge Symmetry and Adjoint Photonic Projection

Let X be four-dimensional spacetime equipped with a smooth Lorentzian metric $g_{\mu\nu}$. We define the ten-dimensional bulk manifold as the product space:

$$Y = X \times K \tag{1}$$

equipped with canonical projection $\pi : Y \rightarrow X$, where K is a compact, connected Calabi-Yau threefold ($\dim_{\mathbb{C}} K = 3$, $\dim_{\mathbb{R}} K = 6$) with $c_1(K) = 0$ and $\text{Hol}(K) = SU(3)$.

2.1 The Symmetry Breaking Cascade

Let $P_{E_8} \rightarrow Y$ be a holomorphic principal $E_8 \times E_8$ bundle over Y . The standard embedding of the spin connection into an $SU(3)$ subgroup of the visible E_8 factor breaks the gauge symmetry to the commutant E_6 :

$$E_8 \longrightarrow SU(3)_K \times E_6. \tag{2}$$

The subsequent reduction of E_6 to the Grand Unified group $SO(10)$ and the Standard Model gauge group $G_{EW} = SU(2)_L \times U(1)_Y$ is modeled as an additional symmetry-breaking cascade realized via discrete Wilson lines associated with $\pi_1(K)$ and bulk Higgs condensation:

$$E_6 \xrightarrow{\text{Wilson lines}} SO(10) \times U(1)_{\psi} \xrightarrow{\text{Higgsing}} SU(3)_C \times SU(2)_L \times U(1)_Y. \tag{3}$$

Definition 1 (Adjoint Electroweak Sheaf). *Let $\text{ad}(P_{E_8}) = P_{E_8} \times_{\text{Ad}} \mathfrak{e}_8^{\mathbb{C}}$ denote the holomorphic adjoint vector bundle on Y . Following the symmetry breaking cascade to $G_{EW} = SU(2)_L \times U(1)_Y$, we define:*

1. *The electroweak adjoint coherent sheaf $\mathcal{E}_{EW} = \mathcal{O}_Y(\text{ad}(P_{EW})) \subset \mathcal{O}_Y(\text{ad}(P_{E_8})) \in \text{Coh}(Y)$ [1].*
2. *The smooth connection 1-form $\mathcal{A}_{EW} = \mathcal{W}^a T^a + \mathcal{B} Y_W \in \Omega^1(Y, \mathfrak{su}(2) \oplus \mathfrak{u}(1))$ with curvature 2-form $\mathcal{F}_{EW} = d\mathcal{A}_{EW} + \mathcal{A}_{EW} \wedge \mathcal{A}_{EW} \in \Omega^2(Y, \mathfrak{g}_{EW})$ satisfying the non-abelian Bianchi identity:*

$$D_{EW} \mathcal{F}_{EW} = d\mathcal{F}_{EW} + [\mathcal{A}_{EW}, \mathcal{F}_{EW}] = 0. \quad (4)$$

3. *The bulk Higgs sheaf $\mathcal{H} = \mathcal{O}_Y(\mathcal{V}_H) \in \text{Coh}(Y)$, where $\mathcal{V}_H$ transforms in the fundamental representation of $SU(2)_L$ with hypercharge $Y_W = 1/2$ [1].*

2.2 Photonic Stabilizer Line Subbundle

Spontaneous symmetry breaking occurs when the bulk Higgs section develops a non-vanishing vacuum expectation value:

$$\langle \mathcal{H} \rangle = \begin{pmatrix} 0 \\ \frac{v_B}{\sqrt{2}} \end{pmatrix}, \quad v_B \in \mathbb{R}^+. \quad (5)$$

This vacuum profile leaves invariant the $U(1)_{\text{em}}$ subgroup generated by the electric charge operator $Q = T^3 + Y_W$.

Definition 2 (Photonic Projection Sheaf). *The unbroken electromagnetic gauge direction is realized as the trivial holomorphic line subbundle $\mathcal{L}_{\gamma}^{\text{ad}} = Y \times \text{span}_{\mathbb{C}}\{Q\} \subset \text{ad}(P_{EW})$. We define the coherent photonic projection sheaf:*

$$\mathcal{E}_{\gamma} = \mathcal{O}_Y(\mathcal{L}_{\gamma}^{\text{ad}}) \in \text{Coh}(Y)1001[1]. \quad (6)$$

The connection on $\mathcal{L}_{\gamma}^{\text{ad}}$ is the abelian physical photon potential $\mathcal{A}_{\gamma} = \sin \theta_W \mathcal{W}^3 + \cos \theta_W \mathcal{B}$, with field strength $\mathcal{F}_{\gamma} = d\mathcal{A}_{\gamma} \in \Omega_{\text{cl}}^2(Y)$ [1].

3 Product Calabi-Yau Geometry and Horizontal-Vertical Bidegrees

For an irreducible Calabi-Yau threefold K with holonomy $\text{Hol}(K) = SU(3)$, the Hodge numbers satisfy $h^{1,0}(K) = 0$, implying that the first Betti number vanishes:

$$b_1(K) = 2h^{1,0}(K) = 0. \quad (7)$$

Consequently, K supports no non-trivial harmonic 1-forms. On the product manifold $Y = X \times K$, the exterior algebra decomposes canonically into horizontal and vertical forms:

$$\Omega^n(Y) = \bigoplus_{p+q=n} \Omega^{p,q}(X \times K) = \bigoplus_{p+q=n} \Omega^p(X) \otimes \Omega^q(K). \quad (8)$$

Fiber integration along K is the degree-shifting linear map $\pi_* : \Omega^{p,q}(X \times K) \rightarrow \Omega^{p+q-6}(X)$ defined by:

$$\pi_*(\alpha_p(X) \wedge \beta_q(K)) = \begin{cases} \alpha_p(X) \int_K \beta_6(K), & q = 6, \\ 0, & q < 6. \end{cases} \quad (9)$$

3.1 Construction of the Differential 9-Form Representative $\mathcal{J}$

Let $\nu_K \in \Omega_{\text{cl},\mathbb{Z}}^6(K)$ denote a normalized closed integral differential form representing the fundamental cohomology class $[K] \in H^6(K, \mathbb{Z})$ satisfying:

$$\int_K \nu_K = 1. \quad (10)$$

Let $\omega_3(X) \in \Omega_{\text{cl},\mathbb{Z}}^3(X)$ denote a closed integral 3-form on physical spacetime representing a class in Cheeger-Simons differential cohomology $\widehat{H}^3(X)$ [2].

Geometric Postulate 1 (Cohomological Representative of Bulk Coupling). *We postulate that the bulk photon-Higgs topological coupling is represented in de Rham cohomology by an integral, closed differential 9-form:*

$$\mathcal{J} = \mathcal{J}_{(3,6)} + \sum_{p>3} \mathcal{J}_{(p,9-p)} \in \Omega_{\text{cl},\mathbb{Z}}^9(Y) 1001[1] \quad (11)$$

whose leading (3,6) horizontal-vertical component is given explicitly by:

$$\mathcal{J}_{(3,6)} = \omega_3(X) \wedge \nu_K. \quad (12)$$

Theorem 1 (Closure and Pushforward Quantization). *Let $\mathcal{J}_{(3,6)} = \omega_3(X) \wedge \nu_K$. Then:*

$$d_Y \mathcal{J}_{(3,6)} = 01001[1]. \quad (13)$$

Proof. Applying the exterior derivative $d_Y = d_X + d_K$:

$$d_Y \mathcal{J}_{(3,6)} = (d_X \omega_3) \wedge \nu_K + (-1)^3 \omega_3 \wedge (d_K \nu_K). \quad (14)$$

Since $\omega_3(X)$ is closed on spacetime ($d_X \omega_3 = 0$) and ν_K is closed on the Calabi-Yau manifold ($d_K \nu_K = 0$), both terms vanish identically:

$$d_Y \mathcal{J}_{(3,6)} = 0 + 0 = 01001[1]. \quad (15)$$

Because $\omega_3(X)$ and ν_K have integral periods, $\mathcal{J}_{(3,6)} \in \Omega_{\text{cl},\mathbb{Z}}^9(Y)$ [1, 2]. $\square$

Remark 1 (Local Gauge Realization on Parity-Conserving Backgrounds). *On physical electromagnetic configurations where the 4D topological charge density vanishes ($F \wedge F = 0$), the abelian Chern-Simons form $\text{cs}_3(A) = A \wedge F$ satisfies $d_X \text{cs}_3(A) = F \wedge F = 0$, supplying an explicit local gauge realization $\omega_3(X) = A \wedge F$.*

4 Fiber Integration and Closed Spacetime Current Dynamics

Pushforward integration of the bulk form $\mathcal{J}$ along the Calabi-Yau threefold K induces the physical soul current on spacetime X [1].

Definition 3 (Spacetime Soul Current). *The physical 3-form soul current J_s on X is defined by:*

$$J_s := \pi_* \mathcal{J} = \int_K \mathcal{J}_{(3,6)} = \omega_3(X) \int_K \nu_K = \omega_3(X) \in \Omega^3(X) 1001[1]. \quad (16)$$

Theorem 2 (Exact Current Conservation). *Because $\partial K = \emptyset$ and $d_Y \mathcal{J} = 0$, the spacetime soul current is strictly closed:*

$$d_X J_s = 01001[1]. \quad (17)$$

Proof. Applying the fiberwise Stokes theorem on the product space with boundaryless fiber K ($\partial K = \emptyset$):

$$d_X J_s = d_X (\pi_* \mathcal{J}) = \pi_* (d_Y \mathcal{J}) + (-1)^{\deg(\mathcal{J})-6} \int_{\partial K} \mathcal{J} = \pi_*(0) + 0 = 01001[1]. \quad (18)$$

$\square$

4.1 Modified Electrodynamics and Homological Boundary Charge

The physical electromagnetic field strength $F \in \Omega^2(X)$ satisfies the extended field equations:

$$dF = 01001[1], \quad (19)$$

$$d * F = J_s + eJ_{\text{em}}1001[1, 4]. \quad (20)$$

Theorem 3 (Gauss-Law Homological Soul Charge). *For any compact spatial 3-chain with boundary $\Sigma^3 \subset X$, the enclosed Gauss-law soul charge is:*

$$Q_{\text{soul}}(\Sigma^3) = \int_{\Sigma^3} J_s = \int_{\partial\Sigma^3} *F - e \int_{\Sigma^3} J_{\text{em}}1001[1]. \quad (21)$$

In charge-neutral regions ($J_{\text{em}} = 0$), $Q_{\text{soul}}(\Sigma^3) = \int_{\partial\Sigma^3} *F$ provides a homological boundary flux invariant that remains invariant under continuous spatial deformations of Σ^3 fixing $\partial\Sigma^3$ [1].

Proof. Integrating $d * F = J_s + eJ_{\text{em}}$ over Σ^3 and applying Stokes' theorem on X :

$$\int_{\Sigma^3} J_s = \int_{\Sigma^3} (d * F - eJ_{\text{em}}) = \int_{\partial\Sigma^3} *F - e \int_{\Sigma^3} J_{\text{em}}1001[1]. \quad (22)$$

□

5 Three Generations on the Tian-Yau Quotient Calabi-Yau Threefold

In $E_8 \times E_8$ heterotic compactifications on Calabi-Yau threefolds, chiral matter fermions arise from the branching of the adjoint representation **248** of E_8 under $SU(3)_K \times E_6$:

$$\mathbf{248} \longrightarrow (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \mathbf{27}) \oplus (\mathbf{8}, \mathbf{1}). \quad (23)$$

The four-dimensional chiral matter fields transform in the fundamental **27** representation of E_6 , which contains a complete Standard Model fermion generation within the **16** of $SO(10) \subset E_6$.

Theorem 4 (Atiyah-Singer Index on Calabi-Yau Threefolds). *Let K be a compact Calabi-Yau threefold with tangent bundle TK , and let $\mathcal{V}_F \rightarrow K$ be an $SU(3)$ vector bundle satisfying $c_1(\mathcal{V}_F) = 0$. The net number of chiral generations $N_{\text{gen}} = n_{27} - n_{\bar{27}}$ is given by the Atiyah-Singer index of the internal Dirac operator $\mathcal{D}_K$:*

$$N_{\text{gen}} = \text{ind}(\mathcal{D}_K) = \int_K \hat{A}(TK) \wedge \text{ch}(\mathcal{V}_F). \quad (24)$$

Proof. On a Calabi-Yau threefold, $c_1(K) = 0$, so $\hat{A}(TK) = 1 + \frac{1}{24}c_2(TK)$. With $c_1(\mathcal{V}_F) = 0$, the Chern character expands as $\text{ch}(\mathcal{V}_F) = \text{rk}(\mathcal{V}_F) - c_2(\mathcal{V}_F) + \frac{1}{2}c_3(\mathcal{V}_F)$. The top degree-6 component is:

$$\left[\hat{A}(TK) \wedge \text{ch}(\mathcal{V}_F) \right]_6 = \frac{1}{2}c_3(\mathcal{V}_F). \quad (25)$$

Integrating over K :

$$N_{\text{gen}} = \frac{1}{2} \int_K c_3(\mathcal{V}_F). \quad (26)$$

□

Proposition 1 (Net Three-Generation Index on the Tian-Yau Threefold). *Let $\tilde{K} \subset \mathbb{P}^3 \times \mathbb{P}^3$ be the complete intersection Calabi-Yau threefold defined as the zero locus of three generic polynomials f_1, f_2, g with multidegrees $(3, 0)$, $(0, 3)$, and $(1, 1)$, satisfying $c_1(\tilde{K}) = (4 - 3 - 0 - 1)H_1 + (4 - 0 - 3 - 1)H_2 = 0$ and Euler characteristic $\chi(\tilde{K}) = -18$.*

The manifold admits a freely acting discrete symmetry group $G \cong \mathbb{Z}_3$. The smooth quotient manifold:

$$K = \tilde{K}/\mathbb{Z}_3 \quad (27)$$

has Euler characteristic:

$$\chi(K) = \frac{\chi(\tilde{K})}{|\mathbb{Z}_3|} = \frac{-18}{3} = -6. \quad (28)$$

Under the standard embedding $\mathcal{V}_F \cong TK$, the Atiyah-Singer index yields:

$$N_{\text{gen}} = \frac{1}{2} \int_K c_3(TK) = \frac{1}{2} |\chi(K)| = \frac{1}{2} |-6| = 3. \quad (29)$$

This establishes a net chiral generation index of three ($n_{27} - n_{\overline{27}} = 3$). The reduction of E_6 to $SO(10)$ and subsequently to the Standard Model gauge group, as well as the lifting of non-chiral vector-like pairs, is achieved via discrete Wilson lines associated with the non-trivial fundamental group $\pi_1(K) \cong \mathbb{Z}_3$.

6 Ten-Dimensional Green-Schwarz Anomaly Matching and Inflow

In ten spacetime dimensions, chiral fermions generate one-loop gauge and gravitational hexagon anomalies characterized by a formal 12-form anomaly polynomial $I_{12}(\mathcal{F}, \mathcal{R})$.

6.1 Trace Conventions and Anomaly Factorization

We adopt the standard Green-Schwarz trace conventions: Tr denotes the trace in the adjoint representation **248** of E_8 , while tr denotes the trace in the fundamental vector representation of $SO(1, 9)$. For E_8 , the fourth-order Casimir invariant satisfies the algebraic identity:

$$\text{Tr } \mathcal{F}^4 = \frac{1}{100} (\text{Tr } \mathcal{F}^2)^2. \quad (30)$$

For the full $E_8 \times E_8$ bundle with curvature $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$, the irreducible 12-form anomaly vanishes identically ($I_{12}^{\text{irr}} \equiv 0$), and the anomaly polynomial factorizes as:

$$I_{12}(\mathcal{F}, \mathcal{R}) = X_4 \wedge X_8 \quad (31)$$

where

$$X_4 = \text{tr } \mathcal{R}^2 - \frac{1}{30} \text{Tr } \mathcal{F}^2, \quad (32)$$

$$X_8 = \frac{1}{24} \text{Tr } \mathcal{F}^4 - \frac{1}{7200} (\text{Tr } \mathcal{F}^2)^2 - \frac{1}{240} \text{Tr } \mathcal{F}^2 \text{tr } \mathcal{R}^2 + \frac{1}{8} \text{tr } \mathcal{R}^4 + \frac{1}{32} (\text{tr } \mathcal{R}^2)^2. \quad (33)$$

6.2 Green-Schwarz Mechanism and Anomaly Inflow

The Kalb-Ramond 2-form field B_2 has modified 3-form field strength $H_3 = dB_2 - \frac{\alpha'}{4}(\omega_{3Y} - \omega_{3L})$, satisfying the Bianchi identity:

$$dH_3 = \frac{\alpha'}{4} \left(\text{tr } \mathcal{R}^2 - \frac{1}{30} \text{Tr } \mathcal{F}^2 \right) = \frac{\alpha'}{4} X_4. \quad (34)$$

Under local gauge and Lorentz transformations with parameter Λ , the 2-form transforms as $\delta_\Lambda B_2 = \frac{\alpha'}{4}(\omega_{2Y}^1 - \omega_{2L}^1)$.

Theorem 5 (Bulk Anomaly Cancellation). *Under the Green-Schwarz factorization $I_{12} = X_4 \wedge X_8$, the classical variation of the Green-Schwarz counterterm:*

$$S_{\text{GS}} = \frac{1}{2(2\pi)^5 \alpha'} \int_Y B_2 \wedge X_8 \quad (35)$$

cancels the factorized 1-loop quantum effective action anomaly $\delta_\Lambda \Gamma_{1\text{-loop}} = - \int_Y I_{10}^{(1)}$:

$$\delta_\Lambda (\Gamma_{1\text{-loop}} + S_{\text{GS}}) = 0. \quad (36)$$

Proof. Using the descent relations $I_{12} = dI_{11}^{(0)}$ and $\delta_\Lambda I_{11}^{(0)} = dI_{10}^{(1)}$, the classical variation of S_{GS} under $\delta_\Lambda B_2 = \frac{\alpha'}{4} d^{-1} X_4^{(1)}$ satisfies:

$$\delta_\Lambda S_{\text{GS}} = \frac{1}{2(2\pi)^5 \alpha'} \int_Y \delta_\Lambda B_2 \wedge X_8 = \int_Y I_{10}^{(1)}. \quad (37)$$

Adding $\delta_\Lambda \Gamma_{1\text{-loop}} + \delta_\Lambda S_{\text{GS}} = 0$ confirms local anomaly matching. $\square$

EFT Hypothesis 1 (Soul Current as Anomaly Inflow). *We hypothesize that the physical 3-form current $J_s = \pi_* \mathcal{J}$ corresponds to the fiber pushforward of the dual anomaly current density $\mathcal{J}_{(3,6)} = X_8 \wedge \nu_K$:*

$$J_s = \pi_* \mathcal{J} = \int_K X_8 \wedge \nu_K \in \Omega^3(X)_{1001}[1]. \quad (38)$$

Under this correspondence, J_s acts as the low-energy boundary inflow current balancing bulk topological anomalies on physical spacetime [1].

7 Effective Action and Precision Phenomenology

Integrating out massive Kaluza-Klein modes of the Calabi-Yau threefold yields the four-dimensional low-energy effective field theory on X [1].

Definition 4 (Low-Energy Effective Action). *The gauge-invariant effective action governing physical electromagnetism and the neutral Z^0 boson is:*

$$S_{\text{eff}}[A, Z] = \int_X \left(-\frac{1}{2} F \wedge *F - \frac{1}{2} Z_2 \wedge *Z_2 - \frac{1}{2} M_Z^2 Z \wedge *Z + A \wedge J_s + \tan \theta_W Z \wedge J_s + eA \wedge J_{\text{em}} + Z \wedge J_Z \right)_{1001} \quad (39)$$

where $Z_2 = dZ \in \Omega^2(X)$, and $J_{\text{em}}, J_Z \in \Omega^3(X)$ are conventional 3-form matter currents.

Proposition 2 (Exact $U(1)_{\text{em}}$ Gauge Invariance). *$S_{\text{eff}}[A, Z]$ is strictly invariant under $A \mapsto A + d\lambda$ for any gauge parameter $\lambda \in C_c^\infty(X)$ assuming matter current conservation ($d_X J_{\text{em}} = 0$).*

Proof. Under $A \mapsto A + d\lambda$, the variation of the electromagnetic interaction terms is:

$$\delta_\lambda S_{\text{eff}} = \int_X d\lambda \wedge (J_s + eJ_{\text{em}}) = \int_X d(\lambda(J_s + eJ_{\text{em}})) - \int_X \lambda d(J_s + eJ_{\text{em}}). \quad (40)$$

By Stokes' theorem on X , the boundary term vanishes for compactly supported λ . Because $d_X J_s = 0$ by Theorem 2 and $d_X J_{\text{em}} = 0$, the bulk integral vanishes identically:

$$\delta_\lambda S_{\text{eff}} = 0. \quad (41)$$

Hence, the topological current coupling preserves exact $U(1)_{\text{em}}$ gauge invariance [1]. $\square$

7.1 Quantum Optical Visibility Attenuation

In a multi-path quantum interferometer enclosing a spatial 3-chain V_3 , single-photon states acquire a geometric phase shift $\Delta\phi_s = \int_{V_3} J_s$ [1]. For a Calabi-Yau threefold with compactification scale $\Lambda = 1/R$ ($k = 6$), the internal volume scales as $\text{Vol}(K) \sim \Lambda^{-6}$. In the presence of stochastic bulk vacuum fluctuations with zero mean and variance $\langle(\Delta\phi_s)^2\rangle \sim \|\omega(\eta)\|^2 \Lambda^{-12}$, the interference visibility $V = |\langle e^{i\Delta\phi_s} \rangle|$ expands to leading order as [1]:

$$1 - V \approx \frac{1}{2} \langle(\Delta\phi_s)^2\rangle = C_V \|\omega(\eta)\|^2 \Lambda^{-12} 1001 [1] \quad (42)$$

where C_V is an $\mathcal{O}(1)$ geometric form factor determined by the triple intersection numbers of K . Ultra-stable optical interferometers with phase sensitivity $\delta\phi \sim 10^{-10} \text{ rad}/\sqrt{\text{Hz}}$ place direct experimental bounds on $\|\omega(\eta)\|$ for $\Lambda \sim 1\text{--}10 \text{ TeV}$.

7.2 Electroweak Radiative Corrections

Topological neutral-current mixing induces a scale-dependent shift in the effective weak mixing angle:

$$\sin^2 \theta_W^{\text{eff}}(q^2) = \sin^2 \theta_W^{\text{SM}}(q^2) + \frac{\sin \theta_W \cos \theta_W \tan \theta_W}{M_Z^2 - q^2} \left(\frac{\|\omega(\eta)\|}{\Lambda^6} \right)^2 1001 [1]. \quad (43)$$

At loop level, vacuum polarization diagrams generate a shift in the muon anomalous magnetic moment:

$$\Delta a_\mu = a_\mu^{\text{exp}} - a_\mu^{\text{SM}} \approx \frac{\alpha}{2\pi} C_\mu \left(\frac{m_\mu}{\Lambda} \right)^2 \left(\frac{\|\omega(\eta)\|}{\Lambda^6} \right)^2 1001 [1]. \quad (44)$$

Furthermore, neutral-current mixing shifts the weak nuclear charge of Cesium (^{133}Cs):

$$\Delta Q_W(^{133}\text{Cs}) = -2 [(2Z + N)\Delta \sin^2 \theta_W] \approx -376 \Delta_{\text{topo}}(q^2 \approx 0) \quad (45)$$

establishing correlated experimental signatures across atomic parity violation, precision quantum optics, and collider regimes [1].

8 Discussion and Grand Unified Outlook

The Grand Unified embedding presented in this work resolves key foundational questions:

1. **Adjoint Gauge Geometry:** By embedding $\mathcal{E}_\gamma = \mathcal{O}_Y(\mathcal{L}_\gamma^{\text{ad}})$ within $\text{ad}(P_{E_8})$, physical electromagnetism is formulated as the surviving generator $Q = T^3 + Y_W$ of a unified $E_8 \rightarrow E_6 \rightarrow SO(10)$ gauge cascade with explicit compactification breaking steps [1].
2. **Exact Three Generations:** On the Tian-Yau complete intersection threefold $K = \tilde{K}/\mathbb{Z}_3$, the Atiyah-Singer index theorem guarantees $\text{ind}(\mathcal{D}_K) = 3$, providing the topological origin of the three fermion families [1].
3. **Cohomological Consistency:** Utilizing the horizontal-vertical decomposition $\Omega^{3,6}(X \times K)$, the pushforward current $J_s = \pi_* \mathcal{J} = \omega_3(X) \in \Omega^3(X)$ is strictly closed ($d_X J_s = 0$), preserving exact 4D gauge invariance [1].

9 Conclusion

We have presented a cohomological formulation and Grand Unified embedding framework for topological photonic sheaves on ten-dimensional spacetime $Y = X \times K$ ($k = 6$) [1]. By deploying horizontal-vertical differential algebra $\Omega^{3,6}(X \times K)$, we formulated the closed integral 9-form

representative $\mathcal{J}_{(3,6)} = \omega_3(X) \wedge \nu_K \in \Omega_{\text{cl},\mathbb{Z}}^9(Y)$ and derived its pushforward 3-form soul current $J_s = \pi_* \mathcal{J} = \omega_3(X) \in \Omega^3(X)$ with $d_X J_s = 0$ [1].

Embedding this structure into $E_8 \times E_8$ heterotic string compactifications establishes:

1. 10D Green-Schwarz anomaly matching via factorization $I_{12} = X_4 \wedge X_8$ under consistent adjoint trace normalizations, supporting the identification of J_s as an anomaly inflow current [1].
2. A net chiral generation index $\text{ind}(\mathcal{D}_K) = n_{27} - n_{\overline{27}} = \frac{1}{2}|\chi(K)| = 3$ on the Tian-Yau quotient Calabi-Yau threefold with $\chi(K) = -6$.
3. Testable, correlated phenomenological predictions across optical interference visibility ($1 - V \sim \Lambda^{-12}$), weak mixing angle shifts, and $(g - 2)_\mu$ [1].

This establishes higher-dimensional photonic continuity as a mathematically grounded, anomaly-consistent foundation for Grand Unified physics [1].

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