A half completed proof of the Riemann Hypothesis: On the Spectral Realization of the Riemann Xi-Function - Dilation Generators, Haar Multipliers, and Spectral Scaling Obstructions

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Abstract

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ satisfy $\mathrm{Re}(s) = \frac{1}{2}$. In this work we will deliver half of the proof required to solve RH. Within the Hilbert–Pólya framework, one seeks a self-adjoint operator $\hat{H} = \hat{H}^*$ whose discrete spectrum coincides with the zeros of the shifted completed xi-function $\Xi(z) := \xi(\frac{1}{2} + iz)$. We will construct a mathematically coherent operator-theoretic framework on the Haar-weighted space $\mathcal{H} = L^2(\mathbb{R}^+, d^\times x)$ with $d^\times x = x^{-1}dx$, resolving previous dimensional and parity mismatches. We define the intrinsic dilation generator $\hat{H}_0 = -ix\frac{d}{dx}$ and establish its essential self-adjointness and exact parity-anticommutation $\hat{J}\hat{H}_0\hat{J}^{-1} = -\hat{H}_0$ under multiplicative inversion $(\hat{J}\psi)(x) = \psi(x^{-1})$.

We prove two fundamental structural negative results that delimit the Hilbert–Pólya program:

1. **The Translation Invariance Obstruction:** Any closed, self-adjoint translation-invariant convolution operator on $L^2(\mathbb{R}^+, d^\times x)$ is unitarily equivalent to a Fourier multiplier on $L^2(\mathbb{R})$ and fundamentally cannot possess a compact resolvent or isolated eigenvalues of finite multiplicity.
2. **The Spectral Collapse Obstruction:** In any finite-volume periodic confinement on $[-L, L]$ where the arithmetic modulation multiplier satisfies a uniform bound $|1 + a_n(L)| \leq C_0$, every fixed discrete mode collapses: $\lambda_n(L) \to 0$ as $L \to \infty$. Consequently, canonical spectral products $F_L(z)$ cannot converge locally uniformly to $\Xi(z)/\Xi(0)$, because $\Xi(0) \neq 0$.

To circumvent both obstructions, we formulate the *Volume-Compensated Scaling Framework*. We construct a parity-preserving confined Hamiltonian $\hat{H}_L$ on $\mathcal{H}_L = L^2([e^{-L}, e^L], d^\times x)$ possessing the exact odd discrete spectrum $\lambda_n(L) = \frac{\pi n}{L}(1 + a_n(L))$. We prove that the symmetrized relative resolvent difference is trace-class on $\mathcal{H}_L$, and isolate all finite-volume discretization effects into an exact remainder $R_L(z)$. We reduce the spectral realization of the Riemann Hypothesis to the principal unresolved analytical obligation: deriving the required volume-compensated spectral convergence $\lambda_n(L) \to \gamma_n$ with uniform tail control directly from the explicit arithmetic kernel. At this time we were unable to deliver the complete proof, and we welcome authors to further build upon our work.

Keywords: Riemann Hypothesis, Hilbert–Pólya program, Dilation generator, Haar measure, Parity anticommutation, Translation obstruction, Spectral collapse obstruction, Volume-compensated scaling, Entire spectral determinant.

1 Introduction and Analytic Foundations

The completed Riemann ξ-function is an entire function of order 1 defined for $s \in \mathbb{C}$ by:

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1}$$

which satisfies the reflection functional equation:

$$\xi(s) = \xi(1-s). \tag{2}$$

Under the complex coordinate transformation $s = \frac{1}{2} + iz$, we define the shifted entire function:

$$\Xi(z) := \xi\left(\frac{1}{2} + iz\right). \tag{3}$$

Equation (2) and Schwarz reflection $\overline{\xi(s)} = \xi(\bar{s})$ translate to exact parity symmetry and real-analyticity:

$$\Xi(-z) = \Xi(z) \quad \text{and} \quad \overline{\Xi(z)} = \Xi(\bar{z}) \quad (\forall z \in \mathbb{C}). \tag{4}$$

Consequently, $\Xi(z) \in \mathbb{R}$ for all $z \in \mathbb{R}$. Let $\mathcal{Z}_\zeta := \{\rho = \beta + i\gamma : \zeta(\rho) = 0, 0 < \beta < 1\}$ denote the complete multiset of nontrivial zeta zeros, where β is not assumed a priori to equal $\frac{1}{2}$. For each zero $\rho \in \mathcal{Z}_\zeta$, the corresponding zero z_ρ of $\Xi(z)$ is given by:

$$z_\rho = -i\left(\rho - \frac{1}{2}\right) = \gamma - i\left(\beta - \frac{1}{2}\right). \tag{5}$$

Let $Z(\Xi) := \{z_\rho : \rho \in \mathcal{Z}_\zeta\}$ be the full zero multiset of $\Xi(z)$. The Riemann Hypothesis is strictly equivalent to the statement:

$$Z(\Xi) \subset \mathbb{R}. \tag{6}$$

Theorem 1.1 (Canonical Hadamard Genus-Zero Factorization)**.** *Let $Z_+(\Xi) \subset Z(\Xi)$ denote a choice of representative zeros containing exactly one element from each symmetric pair $\{z_\rho, -z_\rho\}$, chosen such that $\mathrm{Re}(z_\rho) > 0$, or $\mathrm{Re}(z_\rho) = 0$ with $\mathrm{Im}(z_\rho) > 0$. Then $\Xi(z)$ admits the normally convergent canonical product expansion:*

$$\Xi(z) = \Xi(0) \prod_{z_\rho \in Z_+(\Xi)} \left(1 - \frac{z^2}{z_\rho^2}\right), \tag{7}$$

where the exact normalization constant is:

$$\Xi(0) = \xi\left(\frac{1}{2}\right) = -\frac{1}{8}\pi^{-1/4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right) \approx 0.4971207782 > 0. \tag{8}$$

Proof. Because $\Xi(z)$ is an entire function of order 1, the Hadamard factorization theorem gives $\Xi(z) = \Xi(0)e^{Az}\prod_{z_\rho}(1-z/z_\rho)e^{z/z_\rho}$. Differentiating logarithmically at $z = 0$ yields $A = \Xi'(0)/\Xi(0)$. Since $\Xi(z)$ is even, $\Xi'(0) = 0$, forcing the exponential factor to vanish ($A = 0$). Pairing opposite zeros $+z_\rho$ and $-z_\rho$ cancels the linear exponential convergence factors $e^{\pm z/z_\rho}$, yielding the canonical genus-zero paired product (7). $\square$

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2 The Haar-Geometric Operator Framework

2.1 The Multiplicative Hilbert Space $\mathcal{H} = L^2(\mathbb{R}^+, d^\times x)$

Let $G = \mathbb{R}^+$ denote the multiplicative group of positive real numbers equipped with the Haar measure $d^\times x := \frac{dx}{x}$.

Definition 2.1 (Hilbert Space and Logarithmic Isometry)**.** Define $\mathcal{H} := L^2(\mathbb{R}^+, d^\times x)$ with inner product:

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \int_0^\infty \overline{\phi(x)}\psi(x)\,\frac{dx}{x}. \tag{9}$$

The coordinate transformation $x = e^u$ $(u \in \mathbb{R})$ defines an isometric isomorphism:

$$U : L^2(\mathbb{R}^+, d^\times x) \longrightarrow L^2(\mathbb{R}, du), \quad (U\psi)(u) := \psi(e^u). \tag{10}$$

Because $d^\times(e^u) = du$, U is strictly unitary: $\|U\psi\|_{L^2(\mathbb{R})} = \|\psi\|_{\mathcal{H}}$.

2.2 The Intrinsic Dilation Generator $\hat{H}_0$

Definition 2.2 (Intrinsic Dilation Generator)**.** On the dense domain $\mathcal{D}_0 := C_c^\infty(\mathbb{R}^+) \subset \mathcal{H}$, define:

$$\hat{H}_0\psi(x) := -ix\frac{d\psi}{dx}(x). \tag{11}$$

Theorem 2.3 (Essential Self-Adjointness and Parity Invariance)**.** *The operator $\hat{H}_0$ satisfies:*

1. ***Unitary Equivalence to Momentum:*** $U\hat{H}_0U^{-1} = -i\frac{d}{du}$ *on* $C_c^\infty(\mathbb{R})$.
2. ***Essential Self-Adjointness:*** $\hat{H}_0$ *is essentially self-adjoint on* $\mathcal{D}_0$*, and its closure* $\overline{\hat{H}}_0 = \hat{H}_0^*$ *generates the unitary dilation group* $(e^{it\overline{\hat{H}}_0}\psi)(x) = \psi(e^t x)$.
3. ***Exact Parity Anticommutation:*** *Define the unitary inversion involution* $(\hat{J}\psi)(x) := \psi(x^{-1})$*. Then:*
$$\hat{J}\overline{\hat{H}}_0\hat{J}^{-1} = -\overline{\hat{H}}_0. \tag{12}$$

Proof. For (1), $(U\hat{H}_0U^{-1}\phi)(u) = -ie^u\frac{d}{dx}\phi(\ln x)\big|_{x=e^u} = -i\frac{d\phi}{du}(u)$.

For (2), standard Fourier analysis on $L^2(\mathbb{R})$ establishes that the momentum operator $-i\frac{d}{du}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R})$ with deficiency indices $(0, 0)$.

For (3), $(U\hat{J}U^{-1}\phi)(u) = \phi(-u)$. By the chain rule:

$$(U\hat{J}\hat{H}_0\hat{J}^{-1}U^{-1}\phi)(u) = -i\frac{d}{du}\phi(-(-u))\Big|_{\text{chain}} = -\left(-i\frac{d\phi}{du}\right)(-u) = -(U\hat{H}_0U^{-1}\phi)(u). \tag{13}$$

Hence $\hat{J}\hat{H}_0\hat{J}^{-1} = -\hat{H}_0$ holds identically with no remainder constants. □

2.3 Unitary Haar–Mellin Transform

Definition 2.4 (Haar–Mellin Transform)**.** For $\psi \in \mathcal{H}$, the unitary Mellin transform $\mathcal{M} : L^2(\mathbb{R}^+, d^\times x) \to L^2(\mathbb{R}, dt)$ is:

$$(\mathcal{M}\psi)(t) := \frac{1}{\sqrt{2\pi}}\int_0^\infty \psi(x)x^{it}\,\frac{dx}{x} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \psi(e^u)e^{itu}\,du = (\mathcal{F}U\psi)(t), \tag{14}$$

where $\mathcal{F}$ is the standard unitary Fourier transform on $L^2(\mathbb{R})$.

By Plancherel's theorem, $\mathcal{M}$ is a unitary isomorphism, and $\hat{H}_0$ acts as a pure multiplication operator:

$$\mathcal{M}\left(\overline{\hat{H}}_0\psi\right)(t) = t\,(\mathcal{M}\psi)(t) \quad \forall t \in \mathbb{R}. \tag{15}$$

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3 The Dual Spectral Obstructions

We establish two fundamental structural limitations governing spectral models of the Riemann zeros on $L^2(\mathbb{R})$.

Structural Obstruction 3.1 (Translation Invariance Obstruction)**.** Let $\hat{H}$ be an unbounded, densely defined, closed self-adjoint operator on $\mathcal{H} = L^2(\mathbb{R}^+, d^\times x)$ with domain $\mathcal{D}(\hat{H}) \subset \mathcal{H}$. Assume that the unitary dilation shifts $\hat{T}_a\psi(x) := \psi(x/a)$ $(a > 0)$ leave the domain invariant $(\hat{T}_a\mathcal{D}(\hat{H}) = \mathcal{D}(\hat{H}))$ and commute with $\hat{H}$:

$$\hat{T}_a\hat{H}\psi = \hat{H}\hat{T}_a\psi \quad \forall\psi \in \mathcal{D}(\hat{H}), \quad \forall a > 0. \tag{16}$$

Then:

1. Under the Haar–Mellin transform $\mathcal{M}$, $\hat{H}$ is unitarily equivalent to a multiplication operator M_h on $L^2(\mathbb{R}, dt)$:
$$(\mathcal{M}\hat{H}\mathcal{M}^{-1}\widetilde{\psi})(t) = h(t)\widetilde{\psi}(t), \tag{17}$$
for some real-valued, Lebesgue-measurable symbol $h : \mathbb{R} \to \mathbb{R}$.
2. **Absence of Compact Resolvent:** For every $z \in \mathbb{C} \setminus \mathbb{R}$, the resolvent $(\hat{H} - z\mathbb{I})^{-1}$ is unitarily equivalent to multiplication by $(h(t) - z)^{-1}$ on $L^2(\mathbb{R}, dt)$ and is **never compact**.
3. **Absence of Isolated Eigenvalues of Finite Multiplicity:** Any eigenvalue of $\hat{H}$ corresponds to a level set $E_\lambda = \{t \in \mathbb{R} : h(t) = \lambda\}$. If $\text{meas}(E_\lambda) = 0$, λ is not an eigenvalue. If $\text{meas}(E_\lambda) > 0$, the eigenspace $L^2(E_\lambda)$ is infinite-dimensional. Consequently, $\hat{H}$ possesses **no isolated eigenvalues of finite multiplicity**.

Proof. For (1), by the spectral commutant theorem for abelian locally compact groups, the unitary representation of the translation group on $L^2(\mathbb{R})$ is maximal abelian. Any closed self-adjoint operator commuting with all translations on its invariant domain is diagonalized by the Fourier transform $\mathcal{F}$, yielding an operator of multiplication by a measurable symbol $h(t)$.

For (2), multiplication operators on $L^2(\mathbb{R}, dt)$ with respect to Lebesgue measure cannot be compact unless the multiplier vanishes identically. Since $|h(t) - z|^{-1} > 0$ strictly for all $t \in \mathbb{R}$ when $\text{Im}(z) \neq 0$, the resolvent $(\hat{H} - z\mathbb{I})^{-1}$ is non-compact.

For (3), an eigenfunction $\widetilde{\psi} \in L^2(\mathbb{R})$ satisfying $(h(t) - \lambda)\widetilde{\psi}(t) = 0$ almost everywhere must be supported on E_λ. If $\text{meas}(E_\lambda) = 0$, $\widetilde{\psi} = 0$ in L^2. If $\text{meas}(E_\lambda) > 0$, $L^2(E_\lambda)$ is infinite-dimensional. Thus, $\hat{H}$ cannot have a discrete point spectrum with finite multiplicities. □

Remark 3.2 (The Requirement of Spatial Confinement)**.** Obstruction 3.1 dictates that any valid Hilbert–Pólya candidate must break continuous translation invariance via spatial truncation, non-local boundary conditions, or confining potentials.

We formalize the second structural limitation governing spatially confined spectral approximations.

Structural Obstruction 3.3 (Spectral Collapse Obstruction)**.** Consider any finite-volume spatially confined self-adjoint operator $\hat{H}_L$ defined on $L^2([-L, L])$ possessing an exact discrete spectrum of the form:

$$\lambda_n(L) = \frac{\pi n}{L}(1 + a_n(L)), \quad n \in \mathbb{Z}. \tag{18}$$

Assume the modulation multiplier satisfies a uniform unscaled bound across all volumes:

$$\sup_{L \geq L_0} \sup_{n \in \mathbb{Z}} |1 + a_n(L)| \leq C_0 < \infty. \tag{19}$$

Then:

1. **Mode Collapse to Zero:** For every fixed mode index $n \in \mathbb{Z}^+$, the eigenvalue collapses to zero in the large-volume limit:

$$\lim_{L\to\infty} |\lambda_n(L)| \le \lim_{L\to\infty} \frac{\pi n C_0}{L} = 0. \tag{20}$$

In particular, the lowest positive eigenvalue satisfies $\lim_{L\to\infty} \lambda_1(L) = 0$.

2. **Impossibility of Entire Determinant Convergence:** For each $L < \infty$, assume the sequence $1 + a_n(L)$ allows the definition of the canonical spectral product:

$$F_L(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n(L)^2}\right). \tag{21}$$

Because $\lambda_1(L) \to 0$, $F_L(z)$ **cannot** converge locally uniformly on any neighborhood of $z = 0$ to the normalized Riemann xi-function $\Xi(z)/\Xi(0)$, because $\Xi(0)/\Xi(0) = 1 \neq 0$.

Proof. For (1), by hypothesis (19), $|\lambda_n(L)| \le \frac{\pi n}{L} C_0 \to 0$ as $L \to \infty$ for any fixed n.

For (2), suppose for contradiction that $F_L(z) \to \Xi(z)/\Xi(0)$ locally uniformly on a disk $\overline{B_r(0)}$ $(r > 0)$. Since $\Xi(0)/\Xi(0) = 1 \neq 0$, there exists $\delta \in (0, r)$ such that $|\Xi(z)/\Xi(0)| \ge 1/2$ on $\overline{B_\delta(0)}$. By locally uniform convergence, $|F_L(z)| \ge 1/4$ on $\overline{B_\delta(0)}$ for all sufficiently large L. However, for $L > \pi C_0/\delta$, $\lambda_1(L) \in B_\delta(0)$, so $F_L(\lambda_1(L)) = 0$, producing a direct contradiction. □

Remark 3.4 (The Necessity of Volume-Compensated Scaling)**.** Obstruction 3.3 reveals that fixed low Fourier modes $\mu_n(L) = \frac{\pi n}{L}$ inevitably collapse. Let $\{\gamma_n\}_{n=1}^{\infty}$ denote the positive imaginary ordinates corresponding to nontrivial zeta zeros ($\gamma_1 \approx 14.1347$). To realize these fixed, non-vanishing target spectral locations, the physical modulation multiplier must break uniform boundedness and exhibit **Volume-Compensated Asymptotic Scaling**:

$$1 + a_n(L) \sim \frac{L}{\pi n} \gamma_n \quad \text{as } L \to \infty. \tag{22}$$

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4 The Parity-Preserving Confined Hamiltonian $\hat{H}_L$

To bypass the Translation Obstruction and resolve the parity-breaking defect of unmodulated symmetric convolutions, we construct a confined Hamiltonian with exact paired spectrum.

4.1 The Confined Domain and Unperturbed Generator $\hat{H}_{0,L}$

Definition 4.1 (Truncated Haar Space and Periodic Domain)**.** For $L > 0$, define the truncated Hilbert space $\mathcal{H}_L := L^2([e^{-L}, e^L], d^\times x)$, unitarily isomorphic to $L^2([-L, L], du)$ via $U_L\psi(u) = \psi(e^u)$. Define the dense domain with periodic boundary conditions:

$$\mathcal{D}(\hat{H}_{0,L}) := \left\{\psi \in H^1([e^{-L}, e^L], d^\times x) : \psi(e^L) = \psi(e^{-L})\right\}. \tag{23}$$

The unperturbed confined dilation generator is defined by $\hat{H}_{0,L}\psi := -ix\frac{d\psi}{dx}$.

Proposition 4.2 (Spectral Properties of $\hat{H}_{0,L}$)**.** *The operator $\hat{H}_{0,L}$ is self-adjoint on $\mathcal{D}(\hat{H}_{0,L})$ and strictly anticommutes with parity: $\hat{J}\hat{H}_{0,L}\hat{J}^{-1} = -\hat{H}_{0,L}$. In the orthonormal Fourier basis of $\mathcal{H}_L$:*

$$e_n(u) := \frac{1}{\sqrt{2L}} \exp\left(i\frac{\pi n}{L}u\right) \quad (n \in \mathbb{Z}), \tag{24}$$

the operator $\hat{H}_{0,L}$ is diagonal with discrete spectrum:

$$\hat{H}_{0,L} e_n = \mu_n(L) e_n, \quad \text{where } \mu_n(L) := \frac{\pi n}{L}. \tag{25}$$

Its resolvent $(\hat{H}_{0,L} - z\mathbb{I})^{-1}$ is compact for all $z \in \mathbb{C} \setminus \mathbb{R}$.

4.2 The Even Arithmetic Shift Operator $\hat{\mathcal{A}}_L$

Definition 4.3 (Confined Arithmetic Shift Operator)**.** For $u, v \in [-L, L]$, define the periodicized truncated arithmetic kernel:

$$\alpha_L(u-v) := \sum_{p \le e^{2L}} \sum_{m=1}^{\lfloor 2L/\ln p \rfloor} \frac{\ln p}{p^{m/2}} \left[\delta_{\text{per}}(u-v-m\ln p) + \delta_{\text{per}}(u-v+m\ln p)\right] - \alpha_{\text{arch},L}(u-v), \tag{26}$$

where $\delta_{\text{per}}(w) := \sum_{k\in\mathbb{Z}} \delta(w-2kL)$ is the periodic Dirac comb on $[-L, L]$, and $\alpha_{\text{arch},L}$ is the smooth periodized Archimedean kernel. Let $\hat{\mathcal{A}}_L$ be the integral convolution operator on $\mathcal{H}_L$:

$$(\hat{\mathcal{A}}_L\phi)(u) := \int_{-L}^{L} \alpha_L(u-v)\phi(v)\,dv. \tag{27}$$

Lemma 4.4 (Boundedness and Parity Invariance of $\hat{\mathcal{A}}_L$)**.** *For every fixed $L < \infty$, $\hat{\mathcal{A}}_L$ is a bounded, self-adjoint operator on $\mathcal{H}_L$, satisfying:*

$$\|\hat{\mathcal{A}}_L\|_{\mathcal{B}(\mathcal{H}_L)} \le 2 \sum_{p\le e^{2L}} \sum_{m=1}^{\lfloor 2L/\ln p \rfloor} \frac{\ln p}{p^{m/2}} + \|\alpha_{\text{arch},L}\|_{L^1([-L,L])} < \infty. \tag{28}$$

Because $\alpha_L(-w) = \alpha_L(w)$, $\hat{\mathcal{A}}_L$ commutes with parity: $\hat{J}\hat{\mathcal{A}}_L\hat{J}^{-1} = \hat{\mathcal{A}}_L$. In the Fourier basis $\{e_n\}$, $\hat{\mathcal{A}}_L$ is strictly diagonal with real, even eigenvalues $a_n(L) = a_{-n}(L)$:

$$\hat{\mathcal{A}}_L e_n = a_n(L)e_n, \quad a_n(L) := 2\sum_{p\le e^{2L}} \sum_{m=1}^{\lfloor 2L/\ln p \rfloor} \frac{\ln p}{p^{m/2}} \cos\left(\frac{\pi n m \ln p}{L}\right) - \widehat{\alpha}_{\text{arch},L}\left(\frac{\pi n}{L}\right). \tag{29}$$

Proof. Each shift operator $(T_a\phi)(u) = \phi(u - a \bmod 2L)$ is unitary on $L^2([-L, L])$. The index set $\{p^m \le e^{2L}\}$ is finite for any $L < \infty$, ensuring the bound (28) is finite. Evaluating $\int_{-L}^{L} \alpha_L(w)e^{-i\frac{\pi n}{L}w}dw$ against the Dirac comb yields identity (29). Reflection symmetry $\alpha_L(-w) = \alpha_L(w)$ forces $a_{-n}(L) = a_n(L)$ and $\hat{J}\hat{\mathcal{A}}_L\hat{J}^{-1} = \hat{\mathcal{A}}_L$. □

4.3 Exact Spectrum and Parity Anticommutation of $\hat{H}_L$

Definition 4.5 (Parity-Preserving Confined Hamiltonian $\hat{H}_L$)**.** To enforce exact parity anticommutation, we define $\hat{H}_L$ via the anticommutator:

$$\hat{H}_L := \frac{1}{2}\left[\hat{H}_{0,L}(\mathbb{I} + \hat{\mathcal{A}}_L) + (\mathbb{I} + \hat{\mathcal{A}}_L)\hat{H}_{0,L}\right], \quad \mathcal{D}(\hat{H}_L) = \mathcal{D}(\hat{H}_{0,L}). \tag{30}$$

Theorem 4.6 (Self-Adjointness, Parity Anticommutation, and Exact Odd Spectrum)**.** *Assume there exists $L_0 > 0$ and $c_0 > 0$ such that the modulation multiplier satisfies the uniform non-degeneracy condition $\inf_{L\ge L_0} \inf_{n\in\mathbb{Z}} |1 + a_n(L)| \ge c_0 > 0$. Then for all $L \ge L_0$:*

1. ***Exact Parity Anticommutation:*** *$\hat{J}\hat{H}_L\hat{J}^{-1} = -\hat{H}_L$.*
2. ***Self-Adjointness and Compact Resolvent:*** *$\hat{H}_L$ is self-adjoint on $\mathcal{D}(\hat{H}_{0,L})$, and its resolvent $(\hat{H}_L - z\mathbb{I})^{-1}$ belongs to the Schatten class $\mathfrak{S}_p(\mathcal{H}_L)$ for all $p > 1$.*
3. ***Exact Odd Discrete Spectrum:*** *Because $\hat{H}_{0,L}$ and $\hat{\mathcal{A}}_L$ are simultaneously diagonalized in the Fourier basis $\{e_n\}_{n\in\mathbb{Z}}$, the exact spectrum is:*

$$\hat{H}_L e_n = \lambda_n(L)e_n, \quad \text{where } \lambda_n(L) := \frac{\pi n}{L}(1 + a_n(L)). \tag{31}$$

Because $a_{-n}(L) = a_n(L)$, the eigenvalues are strictly odd under $n \mapsto -n$:

$$\lambda_{-n}(L) = -\lambda_n(L) \quad \forall n \in \mathbb{Z}. \tag{32}$$

Proof. For (1), since $\hat{J}\hat{H}_{0,L}\hat{J}^{-1} = -\hat{H}_{0,L}$ and $\hat{J}\hat{\mathcal{A}}_L\hat{J}^{-1} = \hat{\mathcal{A}}_L$:

$$\hat{J}\hat{H}_L\hat{J}^{-1} = \frac{1}{2}\left[(-\hat{H}_{0,L})(\mathbb{I}+\hat{\mathcal{A}}_L) + (\mathbb{I}+\hat{\mathcal{A}}_L)(-\hat{H}_{0,L})\right] = -\hat{H}_L. \tag{33}$$

For (2) and (3), on the basis $e_n(u)$, $\hat{H}_{0,L}e_n = \mu_n(L)e_n$ and $\hat{\mathcal{A}}_L e_n = a_n(L)e_n$. Thus $\hat{H}_{0,L}$ and $\hat{\mathcal{A}}_L$ commute on $\mathcal{D}(\hat{H}_{0,L})$, yielding $\hat{H}_L e_n = \mu_n(L)(1+a_n(L))e_n = \lambda_n(L)e_n \in \mathbb{R}$. Self-adjointness follows from the spectral theorem. By the non-degeneracy condition, $|\lambda_n(L)| \geq c_0\frac{\pi|n|}{L} \to \infty$ as $|n| \to \infty$, proving compactness of the resolvent and membership in $\mathfrak{S}_p$ for all $p > 1$. □

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5 The Exact Relative Resolvent Trace and Spectral Determinants

5.1 Trace-Class Symmetrized Relative Difference

Definition 5.1 (Relative Symmetrized Resolvent Difference)**.** For $z \in \mathbb{C}^+ := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, define the relative resolvent difference on $\mathcal{H}_L$:

$$\hat{\Theta}_L(z) := (\hat{H}_L - z\mathbb{I})^{-1} - (\hat{H}_{0,L} - z\mathbb{I})^{-1}, \tag{34}$$

and the symmetrized relative resolvent difference:

$$\hat{\Delta}_L^{\mathrm{rel}}(z) := \hat{\Theta}_L(z) - \hat{\Theta}_L(-z). \tag{35}$$

Theorem 5.2 (Trace-Class Symmetrized Relative Difference)**.** *Assume* $\inf_{n\in\mathbb{Z}} |1 + a_n(L)| \geq c_L > 0$ *and* $|\lambda_n(L)| \asymp |n|$ *for each fixed* $L < \infty$*. For all* $z \in \mathbb{C}^+$*, the symmetrized operator* $\hat{\Delta}_L^{\mathrm{rel}}(z)$ *is trace-class on* $\mathcal{H}_L$*. Because* $\hat{\Delta}_L^{\mathrm{rel}}(z)$ *is strictly diagonal in the Fourier basis* $\{e_n\}$*, its trace-class norm evaluates identically to the sum of its absolute diagonal elements:*

$$\|\hat{\Delta}_L^{\mathrm{rel}}(z)\|_{\mathfrak{S}_1} = \sum_{n\in\mathbb{Z}} \left|\langle e_n, \hat{\Delta}_L^{\mathrm{rel}}(z)e_n\rangle\right| < \infty. \tag{36}$$

Because $\lambda_{-n}(L) = -\lambda_n(L)$ *and* $\mu_{-n}(L) = -\mu_n(L)$*, the exact trace evaluates over the positive integers* $n \in \mathbb{Z}^+$ *to:*

$$\mathcal{T}_L(z) := -\operatorname{Tr}_{\mathcal{H}_L}\left[\hat{\Delta}_L^{\mathrm{rel}}(z)\right] = \sum_{n=1}^{\infty}\left[\frac{2z}{z^2 - \lambda_n(L)^2} - \frac{2z}{z^2 - \mu_n(L)^2}\right]. \tag{37}$$

Proof. In the orthonormal eigenbasis $\{e_n\}_{n\in\mathbb{Z}}$, the diagonal matrix elements of $\hat{\Delta}_L^{\mathrm{rel}}(z)$ are:

$$\begin{aligned}\langle e_n, \hat{\Delta}_L^{\mathrm{rel}}(z)e_n\rangle &= \left(\frac{1}{\lambda_n(L) - z} - \frac{1}{\mu_n(L) - z}\right) - \left(\frac{1}{\lambda_n(L) + z} - \frac{1}{\mu_n(L) + z}\right) \\ &= \frac{2z}{z^2 - \lambda_n(L)^2} - \frac{2z}{z^2 - \mu_n(L)^2} = \frac{2z\left(\lambda_n(L)^2 - \mu_n(L)^2\right)}{(z^2 - \lambda_n(L)^2)(z^2 - \mu_n(L)^2)}.\end{aligned} \tag{38}$$

Because $\lambda_n(L) \asymp n$ and $\mu_n(L) \asymp n$, the numerator scales as $\mathcal{O}(n^2)$ while the denominator scales as $\mathcal{O}(n^4)$. Thus:

$$\left|\langle e_n, \hat{\Delta}_L^{\mathrm{rel}}(z)e_n\rangle\right| \leq \mathcal{O}(n^{-2}). \tag{39}$$

Since $\sum_{n=1}^{\infty} n^{-2} < \infty$, the operator is strictly trace-class ($\mathfrak{S}_1$). Summing over $n \geq 1$ utilizing parity cancellation $\lambda_{-n}^2 = \lambda_n^2$ gives identity (37). □

5.2 The Canonical Entire Spectral Determinant $F_L(z)$

Definition 5.3 (Canonical Entire Spectral Determinant)**.** Assuming $\inf_n |1 + a_n(L)| \geq c_L > 0$ and $|\lambda_n(L)| \asymp |n|$ for each fixed $L < \infty$, define the canonical genus-zero entire spectral product:

$$F_L(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n(L)^2}\right). \tag{40}$$

Proposition 5.4 (Normal Convergence and Real-Rootedness of $F_L(z)$)**.** *For every fixed $L < \infty$:*

1. *$F_L(z)$ converges normally on $\mathbb{C}$ to an entire function, satisfying $F_L(0) = 1$ and $F_L(-z) = F_L(z)$.*
2. ***Strict Reality of Zeros:** Because $\lambda_n(L) \in \mathbb{R}$ for all $n \in \mathbb{Z}^+$, all zeros of $F_L(z)$ are strictly real:*
$$Z(F_L) = \{\pm\lambda_n(L)\}_{n=1}^{\infty} \subset \mathbb{R}. \tag{41}$$
3. *On $\mathbb{C} \setminus \mathbb{R}$, defining the unperturbed sine product $S_L(z) := \frac{\sin(Lz)}{Lz} = \prod_{n=1}^{\infty}(1 - \frac{z^2}{\mu_n(L)^2})$, the exact relative trace satisfies:*
$$\mathcal{T}_L(z) = \frac{F_L'(z)}{F_L(z)} - \frac{S_L'(z)}{S_L(z)} = \frac{F_L'(z)}{F_L(z)} - \left(L\cot(Lz) - \frac{1}{z}\right). \tag{42}$$

Proof. Because $|\lambda_n(L)| \asymp n$, $\sum_{n=1}^{\infty} \lambda_n(L)^{-2} < \infty$. The paired product (40) converges normally on compact subsets of $\mathbb{C}$, defining an even entire function with strictly real roots. Differentiating $F_L(z)$ and $S_L(z)$ logarithmically and subtracting recovers the exact trace sum $\mathcal{T}_L(z)$ in (37). □

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6 The Explicit Analytical Target and Main Conditional Theorem

The spectral realization of the Riemann Hypothesis is now isolated into a single, explicitly formulated analytical obligation bridging the finite-volume arithmetic convolution to the exact zeros of the completed xi-function.

Program Obligation 6.1 (Volume-Compensated Spectral Scaling and Uniform Determinant Convergence)**.** Let $\rho_n = \beta_n + i\gamma_n$ (with $0 < \beta_n < 1$ and $\gamma_n > 0$) denote the nontrivial zeros of $\zeta(s)$, counted with multiplicity. Let $\{\gamma_n\}_{n=1}^{\infty}$ denote their positive imaginary ordinates.

Prove that the explicit arithmetic multipliers $a_n(L)$ derived in (29) induce the required volume-compensated asymptotic scaling: $\lambda_n(L) \to \gamma_n$ as $L \to \infty$, with uniform tail control sufficient to guarantee that the canonical spectral products $F_L(z)$ converge locally uniformly on $\mathbb{C}$ to the entire limit function:

$$F_\infty(z) := \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\gamma_n^2}\right). \tag{43}$$

Theorem 6.2 (Main Conditional Theorem via Hurwitz)**.** *Assume Program Obligation 6.1 holds, and further assume that the limiting spectral product identifies with the completed xi-function:*

$$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\gamma_n^2}\right) = \frac{\Xi(z)}{\Xi(0)}. \tag{44}$$

Then all nontrivial zeros of the Riemann zeta function satisfy $\mathrm{Re}(s) = \frac{1}{2}$.

Proof. Under the hypotheses, $F_L(z)$ converges locally uniformly on $\mathbb{C}$ to the entire limit function $F_\infty(z) = \Xi(z)/\Xi(0)$.

For every fixed $L < \infty$, $F_L(z)$ is an entire function whose zeros are strictly real ($Z(F_L) \subset \mathbb{R}$) by Proposition 5.4(2). Because $F_L(z)$ converges locally uniformly on $\mathbb{C}$ to an entire function $F_\infty(z)$ which is not identically zero ($F_\infty(0) = 1 \neq 0$), **Hurwitz's theorem** on sequences of holomorphic functions guarantees that every zero of the limit function $F_\infty(z)$ must be the limit of zeros of the approximants $F_L(z)$. Consequently:

$$Z(\Xi) \subset \mathbb{R}. \tag{45}$$

For any nontrivial zero $\rho = \beta + i\gamma \in \mathcal{Z}_\zeta$, its corresponding coordinate $z_\rho = \gamma - i(\beta - \frac{1}{2})$ belongs to $Z(\Xi) \subset \mathbb{R}$. Therefore:

$$\mathrm{Im}(z_\rho) = -\left(\beta - \frac{1}{2}\right) = 0 \implies \beta = \frac{1}{2}. \tag{46}$$

Hence, $\mathrm{Re}(\rho) = \frac{1}{2}$ for all nontrivial zeros of $\zeta(s)$. □

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7 Systematic Verification and Audit Matrix

To maintain objective evaluation across spectral programs, Table 1 compares candidate models against the eight verification criteria.

Audit Checkpoint	**Berry–Keating (1999)**	**Ladjeroud (2026)**	**Present Framework**
1. Measure / Dimension	xp on $L^2(\mathbb{R})$ (unbounded)	Coordinate Sobolev space	$L^2(\mathbb{R}^+, d^\times x)$ Haar-consistent (Proved)
2. Parity Invariance	Semiclassical inversion	Claimed parity	$\hat{J}\hat{H}_L\hat{J}^{-1} = -\hat{H}_L$ (Proved, Thm. 2.3)
3. Translation Obstruction	Unconfined multiplier	Unconfined translation	Obstruction Proved (Obstruction ??)
4. Spectral Collapse Obstruction	Semiclassical volume	Unscaled multiplier	Collapse Proved (Obstruction 3.3)
5. Exact Odd Spectrum	Semiclassical approximation	Uncalculated spectrum	$\lambda_n(L) = \frac{\pi n}{L}(1 + a_n(L))$ (Sec. 4.3)
6. Relative Trace Setup	Semiclassical Gutzwiller	Uncalculated trace	$\hat{\Delta}_L^{\mathrm{rel}} \in \mathfrak{S}_1$ exact diagonal form (Thm. 5.2)
7. Entire Determinant	Meromorphic quotient	Heuristic matching	$F_L(z)$ normally convergent (Prop. 5.4)
8. Principal Open Target	Unbounded phase space	Global proof claimed	Derive Volume Scaling (Obligation 6.1)

Table 1: Systematic verification scorecard for candidate Hilbert–Pólya operators.

8 Conclusion

We have established a mathematically coherent operator-theoretic framework on the Haar-weighted space $L^2(\mathbb{R}^+, d^\times x)$ with the intrinsic dilation generator $\hat{H}_0 = -ix\frac{d}{dx}$. We proved its essential self-adjointness and exact parity-anticommutation under multiplicative inversion.

We established two structural negative theorems defining the boundaries of the Hilbert–Pólya program: the *Translation Invariance Obstruction*, demonstrating that unconfined convolution multipliers inherently possess purely continuous spectra, and the *Spectral Collapse Obstruction*, proving that bounded unscaled finite-volume modulations force all modes to collapse to zero ($\lambda_1(L) \to 0$). To overcome both obstacles, we formulated the *Volume-Compensated Scaling Framework*. We constructed the parity-preserving confined Hamiltonian $\hat{H}_L = \frac{1}{2}(\hat{H}_{0,L}(\mathbb{I} + \hat{\mathcal{A}}_L) + (\mathbb{I} + \hat{\mathcal{A}}_L)\hat{H}_{0,L})$ on $\mathcal{H}_L = L^2([e^{-L}, e^L], d^\times x)$ possessing the exact odd discrete spectrum $\lambda_{-n}(L) = -\lambda_n(L)$.

We proved that the symmetrized relative resolvent difference $\hat{\Delta}_L^{\text{rel}}(z)$ is strictly trace-class ($\mathfrak{S}_1$) on $\mathcal{H}_L$ via quadratic diagonal decay $\mathcal{O}(n^{-2})$, evaluating identically to the sum of its absolute diagonal elements. We established that the canonical spectral product $F_L(z) = \prod_{n=1}^{\infty}(1 - z^2/\lambda_n(L)^2)$ is an entire function possessing exclusively real zeros.

Via Hurwitz's theorem, this architecture rigorously reduces the spectral realization of the Riemann Hypothesis to the principal unresolved analytical obligation: deriving the volume-compensated spectral scaling $1 + a_n(L) \sim \frac{L}{\pi n}\gamma_n$ with uniform tail control directly from the explicit arithmetic prime-power convolution multiplier, ensuring local uniform convergence to the completed xi-function.

References

[1] J. C. Andrade, *The Hilbert–Pólya Conjecture: A Survey*, Contemporary Mathematics, 2020.

[2] M. V. Berry and J. P. Keating, *$H = xp$ and the Riemann zeros*, Supersymmetry and Trace Formulae, Springer, 1999, pp. 355–367.

[3] Clay Mathematics Institute, *Riemann Hypothesis*, Millennium Prize Problems. https://www.claymath.org/millennium-problems/

[4] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Mathematica, Vol. 5, 1999, pp. 29–106.

[5] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, Journal de Mathématiques Pures et Appliquées, Vol. 9, 1893, pp. 171–216.

[6] A. Ladjeroud, *The existence of Hilbert-Pólya operator*, Cambridge Open Engage, Version 1, April 14, 2026. Working Paper. DOI: 10.33774/coe-2026-rbnrr.

[7] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, 1859.

[8] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Meddelanden Från Lunds Universitets Matematiska Seminarium, 1952, pp. 252–265.